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A REMARK ON CLASSIFICATION OF RIEMANN SURFACES WITH RESPECT TO $\Delta u = Pu$

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1. Consider a C^1 differential $P(z)\,dx\,dy$ ($z = x+iy$; $P \geqq 0$, $P \not\equiv 0$) on an open Riemann surface R. We denote by $\mathcal{O}_{PX}$ the set of pairs (R, P) such that the subspace $PX(R)$ of the space $P(R)$ of C^2 solutions u of $\Delta u = Pu$ on R determined by a property X reduces to $\{0\}$. Here the possibilities for X that we consider are B (boundedness), D (Dirichlet-finite: $D_R(u) = \int_R |\text{grad } u(z)|^2 dxdy$), E (energy-finite: $E_R(u) = D_R(u) + \int_R P(z)(u(z))^2 dxdy$), and their combinations BD, BE. The purpose of this note is to announce that the following very simple pair (U, Q) given by

$$(1) \qquad U = \{z; \ |z| < 1\}, \qquad Q(z) = (1 - |z|)^{-1}$$

is an example of the strict inclusion relation

$$(2) \qquad \mathcal{O}_{PD} < \mathcal{O}_{PE}.$$

Here and hereafter $\mathfrak{A} < \mathfrak{B}$ means that $\mathfrak{A}$ is a proper subset of $\mathfrak{B}$. This type of classification problem for Riemann surfaces proposed by

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Ozawa [5] and Royden [6] thus comes to the following complete conclusion:

$$(3) \qquad \mathcal{O}_G < \mathcal{O}_{PB} < \mathcal{O}_{PD} = \mathcal{O}_{PBD} < \mathcal{O}_{PE} = \mathcal{O}_{PBE}$$

for pairs (R, P) of Riemann surfaces R and C^1 differentials P on R ($P \geqq 0$, $P \not\equiv 0$). Here $\mathcal{O}_G$ is the set of pairs (R, P) such that R does not carry a harmonic Green's function. For the latest information on this subject, refer to [1].

2. Since the harmonic Green's function $G(z, \zeta)$ on U is $\log(|1-\bar{\zeta}z|/|z-\zeta|)$, we have

$$\int_0^{2\pi} G(re^{i\theta}, \zeta)\, d\theta = -2\pi \max(\log r, \log |\zeta|).$$

By virtue of this relation it is easy to evaluate

$$(\mathrm{D}) \qquad \int_{U\times U} G(z, \zeta)Q(z)Q(\zeta)\, dx\, dy\, d\xi\, d\eta < \infty \qquad (\zeta = \xi + i\eta).$$

By the integral comparison theorem ([3], [4], [2]), (D) implies the existence of an order-preserving isometric vector space isomorphism $u \rightarrow \tau u$ of $\mathrm{QBD}(U)$ onto $\mathrm{HBD}(U)$ (H stands for harmonic) determined by

$$(4) \qquad u = \tau u - \frac{1}{2\pi}\int_U G(\cdot, \zeta)Q(\zeta)u(\zeta)\, d\xi\, d\eta.$$

In particular we obtain $(U, Q) \notin \mathcal{O}_{PBD}$. Since $\mathcal{O}_{PBD} = \mathcal{O}_{PD}$ ([2], [3]), we conclude that

$$(5) \qquad (U, Q) \notin \mathcal{O}_{PD}.$$

Observe that every $u \in \mathrm{QBE}(U)$ ($\subset \mathrm{QBD}(U)$) is a difference of two nonnegative $u_i \in \mathrm{QBE}(U)$, i.e. $u = u_1 - u_2$ (Royden [6]). Let $u \in \mathrm{QBE}(U)$ and $u \geqq 0$. Since

$$\int_{|z|<1} (1 - |z|)\, d\mu(z) < \infty \qquad (d\mu(z) = Q(z)u(z)\, dx\, dy \geqq 0),$$

by Littlewood's theorem we have

$$\lim_{r\to 1} \int_U G(re^{i\theta}, \zeta)\, d\mu(\zeta) = 0$$

for almost every $\theta \in [0, 2\pi]$ (cf. e.g. Tsuji [7, p. 170]). As the bounded harmonic function τu has the radial limit $\lim_{r\to 1} \tau u(re^{i\theta})$ for almost every $\theta \in [0, 2\pi]$, we see by (4) that the same is true for u and a fortiori

$$\lim_{r\to 1} u(re^{i\theta}) = \lim_{r\to 1} \tau u(re^{i\theta}) = u^*(\theta) \geqq 0 \tag{6}$$

almost everywhere on $[0, 2\pi]$. If $u^*(\theta) > 0$, then $u(re^{i\theta}) > u^*(\theta)/2$ for $0 < \epsilon < r < 1$ and therefore

$$\begin{aligned} l(\theta) &= \int_0^1 Q(re^{i\theta})(u(re^{i\theta}))^2 r\, dr \\ &\geqq 4^{-1}(u^*(\theta))^2 \int_\epsilon^1 \frac{1}{1-r} r\, dr = \infty. \end{aligned} \tag{7}$$

By Fubini's theorem,

$$\int_U Q(z)(u(z))^2\, dx\, dy = \int_0^{2\pi} l(\theta)\, d\theta. \tag{8}$$

In view of (7), the quantity (8) is finite only if $u^*(\theta) = 0$ almost everywhere on $[0, 2\pi]$. Then by Poisson's representation, we deduce $\tau u \equiv 0$ and consequently by $0 \leqq u \leqq \tau u$ we conclude that $u = 0$. Therefore $u \in QBE(U)$ $(u \geqq 0)$ implies that $u \equiv 0$, i.e. $(U, Q) \in \mathcal{O}_{PBE}$. Since $\mathcal{O}_{PBE} = \mathcal{O}_{PE}$ (Royden [6]), we obtain

$$(U, Q) \in \mathcal{O}_{PE}. \tag{9}$$

The relations (5) and (9) imply (2).

3. In our recent paper [4] (see also [1]) we determined the degeneracy character of (E^m, P_α) (E^m: m-dimensional Euclidean space $(m \geqq 3)$; $P_\alpha(x) \sim |x|^{-\alpha}$ $(|x| \to \infty)$) as follows:

$$\begin{aligned} &(E^m, P_\alpha) \in \mathcal{O}_{PB} - \mathcal{O}_G && (\alpha \leqq 2); \\ &(E^m, P_\alpha) \in \mathcal{O}_{PD} - \mathcal{O}_{PB} && (2 < \alpha \leqq (m+2)/2); \\ &(E^m, P_\alpha) \in \mathcal{O}_{PE} - \mathcal{O}_{PD} && ((m+2)/2 < \alpha \leqq m); \\ &(E^m, P_\alpha) \notin \mathcal{O}_{PE} && (m < \alpha). \end{aligned} \tag{10}$$

The 2-dimensional analogue is (U, P_α) $(P_\alpha(z) \sim (1-|z|)^{-\alpha}$ $(|z| \to 1))$: The pair (U, P_α) will be an example for each strict inclusion in (3) if α is properly chosen, which will be discussed in detail elsewhere.

ADDED IN PROOF. The 2-dimensional analogue of (10) for (U, P_α) is: $2 \leqq \alpha$; $3/2 \leqq \alpha < 2$; $1 \leqq \alpha < 3/2$; $\alpha < 1$ (M. Nakai, *The equation $\Delta u = Pu$ on the unit disk with almost rotation free $P \geqq 0$* (to appear)).

REFERENCES

1. M. Glasner, R. Katz and M. Nakai, *A remark on classification of Riemannian manifolds with respect to* $\Delta u = Pu$, Bull. Amer. Math. Soc. **77** (1971), 425–428.

2. M. Nakai, *Dirichlet finite solutions of* $\Delta u = Pu$, *and classification of Riemann surfaces*, Bull. Amer. Math. Soc. **77** (1971), 381–385.

3. ———, *Dirichlet finite solutions of* $\Delta u = Pu$ *on open Riemann surfaces*, Kōdai Math. Sem. Rep. (to appear).

4. ———, *The equation* $\Delta u = Pu$ *on* E^m *with almost rotation free* $P \geqq 0$, Tôhoku Math. J. (to appear).

5. M. Ozawa, *Classification of Riemann surfaces*, Kōdai Math. Sem. Rep., **1952**, 63–76. MR **14**, 462.

6. H. Royden, *The equation* $\Delta u = Pu$ *and classification of open Riemann surfaces*, Ann. Acad. Sci. Fenn. Ser. A I No. 271 (1959). MR **22** #12215.

7. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959. MR **22** #5712.

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