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ON THE SCHUR SUBGROUP

BY K. L. FIELDS[1]

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Let A_x be a simple component of the group algebra QG of a finite group G over the rationals Q having center K. Let p be a rational prime and Q_p the p-adic completion of Q. Then $Q_p \otimes_Q A_x$ is a direct sum of simple algebras $K_{\mathfrak{p}} \otimes_K A_x$ where $\mathfrak{p}$ is a prime divisor of p. The characters associated with these simple components are all conjugate. Since if a representation with K-valued character χ can be written in $K_{\mathfrak{p}}$, then a representation with character χ^σ may be written in $K_{\mathfrak{p}\sigma}$, we see that either A_x is split at every prime dividing p or at none. This fact has also been observed independently by Mark Benard.

We are now in a position to apply results of Mac Lane [**3**, Corollary] together with those of [**1**]:

THEOREM 1. *If A_x is a quaternion algebra central over K, then $A_x \sim K \otimes_L B$ where L is any subfield of K such that the galois group of K/L is cyclic, and B is a simple algebra central over L.*

Hence if K/Q is cyclic of odd order, the simple algebras central over K appearing in some QG are precisely those of the form $K \otimes_Q B$ where B is a quaternion algebra over Q (cf. [**2**]).

As a final remark, we observe that the above together with the construction of [**1**] enable us to determine those algebras of index 3 in Sch $(Q\sqrt{(-3)})$, the Schur subgroup of $Q(\sqrt{(-3)})$. By [**2**], they are *not* of the form $Q(\sqrt{(-3)}) \otimes_Q A$. Hence by Mac Lane [**3**], the above remarks and the following construction, they must have zero Hasse invariant at any primes which are ramified or inertial; i.e. if $p \equiv -1 \bmod (3)$ or $p=3$ then there is zero Hasse invariant at primes of $Q(\sqrt{(-3)})$ extending these. If $p \equiv 1$ (3), then p splits into 2 primes in $Q(\sqrt{(-3)})$, and we show that there is an A_x with Hasse invariant $\frac{1}{3}$ at one of these primes, $\frac{2}{3}$ at the other, and zero elsewhere: Simply let G be the group generated by a, b, c where $a^p=1$, $b^{p-1}=c$, $c^3=1$ and where c is central and b acts on a as the generator of the galois group

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$Q(\xi_p)/Q$ acts on ξ_p. Then QG contains the simple component $\langle K(\xi_p), \tau, \xi_3\rangle$ where $K=Q(\sqrt{(-3)})$. Summarizing,

THEOREM 2. *The simple algebras of index* 3 *central over* $Q(\sqrt{(-3)})$ *which appear in some* QG *are precisely those which do not split and have different Hasse invariants at both extensions of a finite number of* $p \equiv 1$ (3), *and split everywhere else.*

THEOREM 3. *The simple algebras of exponent* 4 *central over* $Q(i)$ *which appear (up to similarity) in some* QG *are precisely those of the form* $A\otimes_Q B$ *where* A *is a quaternion algebra central over* Q, *and* B *is an algebra central over* $Q(i)$ *which has Hasse invariants* 1/4 *and* 3/4 *at the extensions of a finite number of* $p\equiv 1$ (4) *and splits everywhere else.*

ADDED IN PROOF. We construct an algebra in $S(Q\sqrt{10})$ which is not of the form $Q\sqrt{10}\otimes_Q D$ for any quaternion algebra D central over the rationals. This shows that the real quadratic and quadratic imaginary cases are substantially different. Let G be the group generated by x, y, a, b where

$$x^5 = 1, \quad a^8 = 1, \quad y^4 = a^2, \quad b^2 = 1,$$

$$b^{-1}ab = a^{-1}, \quad y^{-1}ay = a^5, \quad y^{-1}by = a^5b, \quad y^{-1}xy = x^2,$$

$$xa = ax, \quad xb = bx.$$

It is clear that QG contains the simple component $A=\langle Q(\xi_5, \xi_8), Z_4\times Z_2, c\rangle$ whose center is $Q\sqrt{10}$, and where the values of c are certain 8th roots of 1. Furthermore, the completion of A at the prime $\mathfrak{z}$ dividing 5 contains the cyclic algebra $B=\langle Q_5(\xi_5), Z_4, i\rangle$ which is of exponent 4. Hence $A_{\mathfrak{z}}\cong B\otimes_{Q_5} C$ where C, the centralizer of B in $A_{\mathfrak{z}}$, is a split 4 dimensional algebra central over $Q_5\sqrt{10}$. In particular A does not split at $\mathfrak{z}$, and so is not of the form $Q\sqrt{10}\otimes_Q D$ for any D of exponent 2.

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UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637