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BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS ON HERMITIAN HYPERBOLIC SPACE[1]

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Let $D=\{z=(z_1,\cdots,z_n)\in C^n:h(z)=\text{Im } z_1-\sum_2^n |z_k|^2>0\}$, and $B=\partial D=\{z:h(z)=0\}$. Writing $z_j=x_j+iy_j$ we let β be the measure on B given by $d\beta=dx_1dx_2dy_2\cdots dx_ndy_n$. D is a Siegel domain of Type II which is the image of the unit ball $D=\{z\in C^n:\sum_1^n |z_k|^2<1\}$ under the generalized Cayley transform:

$$z_1\mapsto i\frac{1+z_1}{1-z_1},\quad z_k\to\frac{iz_k}{1-z_1},\qquad k=2,\cdots,n.$$

Let N be the group of holomorphic automorphisms of D consisting of the elements $(a,c)\in R\times C^{n-1}$ acting on D in the following way:

$$(a,c):z_1\to z_1+a+2i\sum_{k=2}^n z_k\bar{c}_k+i\sum_{k=2}^n |c_k|^2,$$
$$(a,c):z_k\to z_k+c_k,\qquad k\geqq 2.$$

N acts simply transitively on B. We will consider real-valued functions on D which are harmonic with respect to the Laplace-Beltrami operator:

$$L=h(z)\left\{4y_1\frac{\partial^2}{\partial z_1\partial\bar{z}_1}+\sum_2^n\frac{\partial^2}{\partial z_k\partial\bar{z}_k}+2i\sum_2^n\bar{z}_k\frac{\partial^2}{\partial z_1\partial\bar{z}_k}-2i\sum_2^n z_k\frac{\partial^2}{\partial\bar{z}_1\partial z_k}\right\}.$$

In [2] Korányi defined the following notion of admissible convergence in D: let us call

$$\Gamma_\alpha(u)=\left\{z\in D:\text{Max}\left[\,|\text{Re } z_1-\text{Re } u_1|,\ \sum_2^n|z_k-u_k|^2\right]<\alpha h(z),\ h(z)<1\right\}$$

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a truncated admissible domain of aperture α at $u \in B$. We say that f on D converges admissibly at u to l if $\lim_{z\to u;\, z\in\Gamma_\alpha(u)} f(z) = l$, for some $\alpha > 0$.

The principal result of this note is the Theorem below, which is the analogue of results of Marcinkiewicz and Zygmund [3], Spencer [4], Calderón [1], and Stein [5]. (This is often referred to as the Area theorem for harmonic functions.) Let

$$\nabla f = \left(2h^{1/2}\frac{\partial f}{\partial z_1},\ 2i\bar{z}_2\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2},\ \cdots,\ 2i\bar{z}_n\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_n}\right)$$

and

$$|\nabla f|^2 = 4h\left|\frac{\partial f}{\partial z_1}\right|^2 + \sum_2^n \left|2i\bar{z}_k\frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_k}\right|^2.$$

Let E be a measurable set in B and suppose that f is a real-valued harmonic function in D.

THEOREM. (a) *If f is admissibly bounded for each point of E then*

$$\int_{\Gamma_\alpha(u)} h(z)^{-n} |\nabla f|^2 d\mu(z) < \infty \tag{1}$$

for almost every u in E and $\alpha > 0$, where $d\mu$ is Lebesgue measure.

(b) *If, for each point u of E, we can find an α such that the integral* (1) *is finite, then f converges admissibly at almost every point of E.*

The general outline of the proof follows Stein [5]. The differences arise from the fact that the Laplace-Beltrami operator is not uniformly elliptic. We first indicate how part (a) is proved. By a standard argument (see Calderón [1]) we may assume that E is compact, and f is uniformly bounded in $\Gamma_\alpha(u)$, for α fixed, and all $u \in E$.

LEMMA 1. *If f is bounded and harmonic in $\Gamma_\alpha(0)$, then $h(z)|\partial f/\partial z_1|$ and $h(z)^{1/2}|\partial f/\partial z_k|$, $k \geq 2$, are bounded in $\Gamma_{\alpha'}(0)$ for $\alpha' < \alpha$.*

This result can be proved by using the Poisson integral representation for functions defined on images of spheres under the Cayley transform.

Let $\omega_\alpha(E) = \bigcup_{u\in E}\Gamma_\alpha(u)$. We construct regions approximating $\omega_\alpha(E)$. Write $z \in D$ as $z = [x, \tilde{z}]_t$ where $x = x_1$, $\tilde{z} = (z_2, \cdots, z_n)$, $t = h(z)$. Since E is compact, $E_t = \{[x, \tilde{z}]_t : [x, \tilde{z}]_0 \in E\}$ is compact. For $0 < t < 1$ let $\Gamma_\alpha(u)_{t^2} = \{[x, \tilde{z}]_{r+t^2} : [x, \tilde{z}]_r \in \Gamma_\alpha(u)$ and $r + t^2 < 1\}$. Then $\{\Gamma_\alpha(u)_{t^2} \cap E_t\}_{u\in E}$ forms an open cover of E_t. Choose a finite subcover for $t = t_0 < 1$ and then for each $t < t_0$ choose one in the following manner: if $u_1, \cdots, u_{k(t)}$ are the base points chosen for the cover of

E_t, and if $t'<t''<t_0$, then $\{u_1, \cdots, u_{k(t')}\} \supset \{u_1, \cdots, u_{k(t'')}\}$. Let $\omega_t = \bigcup_{j=1}^{k(t)} \Gamma_\alpha(u_j)_{t^2}$.

LEMMA 2. $\int_{\omega_\alpha(E)} |\nabla f|^2 d\mu(z) < \infty$.

We prove this by first applying Green's theorem to ω_t. Then, using the estimates of Lemma 1 translated by the group N and the uniform boundedness of f, we obtain $\int_{\omega_t} |\nabla f|^2 d\mu(z) \leqq k\int_{\partial\omega_t} ds$ when k is independent of t. Now we let t tend to 0, and observe that $\int_{\partial\omega_t} ds \leqq M$ independently of t. Part (a) then follows from:

LEMMA 3. *Suppose $E \subset B$ is compact and f is locally bounded and positive in D. If $\int_{\omega_\alpha(E)} f\, d\mu < \infty$, then $\int_{\Gamma_\beta(u)} h(z)^{-n} f(z)\, d\mu(z) < \infty$ for all $\beta > 0$ and almost every $u \in E$.*

We now outline the proof of part (b).

LEMMA 4. *If $\int_{\Gamma_\alpha(0)} h(z)^{-n} |\nabla f|^2 d\mu(z) < \infty$, then $h(z)|\partial f/\partial z_1|$ and $h(z)^{1/2}|\partial f/\partial z_k|$, $k \geqq 2$, are bounded in $\Gamma_{\alpha'}(0)$ for $\alpha' < \alpha$.*

To prove this let

$$D_1 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar z_1},$$

$$D_k = 2iz_k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar z_k},$$

$$D_{k'} = -2i\bar z_k \frac{\partial}{\partial \bar z_1} + \frac{\partial}{\partial z_k},$$

$$D_0 = z_1 \frac{\partial}{\partial z_1} + \bar z_1 \frac{\partial}{\partial \bar z_1} + \frac{1}{2} \sum_2^n \left(z_k \frac{\partial}{\partial z_k} + \bar z_k \frac{\partial}{\partial \bar z_k} \right).$$

We then observe that if f is harmonic then $D_0 f$, $D_1 f$, $D_k f$, $D_{k'} f$ are harmonic, and thus can be represented as Poisson integrals. Now $|\nabla f|^2$ dominates $h|D_1 f|^2$, $|D_k f|^2$, $|D_{k'} f|^2$ and $h^{-1}|D_0|^2$ in $\Gamma_\alpha(0)$; and the latter dominate $h|\partial f/\partial z_1|^2$ and $|\partial f/\partial z_k|^2$ for $k \geqq 2$, in $\Gamma_\alpha(0)$. Now, using Green's theorem and Lemma 4, we have

$$\int_{\partial\omega_t} f^2 ds \leqq k \int_{\partial\omega_t} |f|\, ds + k' \int_{\omega_t} |\nabla f|^2 d\mu.$$

LEMMA 5. *Suppose $E \subset B$ is compact, f is nonnegative and locally bounded in D, and for each $u \in E$, there exists an $\alpha > 0$ such that $\int_{\Gamma_\alpha(u)} f\, d\mu < \infty$. Then for every $\epsilon > 0$ and $\beta > 0$ there exists a compact set $F \subset E$ such that* meas$(E\backslash F) < \epsilon$, *and $\int_{\partial\omega_\beta(F)} h(z)^n f(z) d\mu(z) < \infty$.*

Applying this to the inequality above we have $\int_{\partial\omega_t} f^2 ds \leqq M$ independently of t. Now a standard argument (see Stein [5]) shows that $|f(z)| \leqq cg(z)+c'$ in $\omega_\alpha(E)$ where g is the Poisson integral of some function in $L^2(B)$. The result now follows from Korányi [2].

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