BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, May 1971

# CLASSIFICATION OF THE SIMPLE SEPARABLE REAL $L^*$-ALGEBRAS[1]

BY IGNACIO UNSAIN

Communicated by M. H. Protter, October 20, 1970

A real (complex) $L^*$-algebra is a Lie algebra $L$ whose underlying vector space is a real (complex) Hilbert space and such that, for each $x \in L$, there exists an $x^* \in L$ satisfying $\langle [x, y], z\rangle = \langle y, [x^*, z]\rangle$ for all $y, z$ in $L$. J. R. Schue [11], [12] defined and classified the simple separable complex $L^*$-algebras. V. K. Balachandran [1], [2], [3], [4], [5] gave a more general setting to the techniques used by Schue for not necessarily separable $L^*$-algebras; he also defined the notions of real form and compact real form.

The main result of this work is the classification of the simple separable real $L^*$-algebras up to $L^*$-isomorphisms. The classification was also obtained, independently, by Mr. Pierre de la Harpe.

The following can be shown:

THEOREM 1. *The complexification $\tilde{L}$ of a simple $L^*$-algebra $L$ is not simple if and only if $L = M^R$, where $M$ is a simple complex $L^*$-algebra ($M^R$ denotes the real $L^*$-algebra obtained from $M$ by restriction of scalars).*

Therefore, the classification reduces essentially, aside from simple $L^*$-algebras having a complex structure which are in a one-to-one correspondence with the simple complex $L^*$-algebras, to the study of the real forms of all simple complex $L^*$-algebras.

If $L$ is a real form of a semisimple $L^*$-algebra $\tilde{L}$, the decomposition $L = K + M$ (Hilbert direct sum), where $K = \{a \in L : a^* = -a\}$ and $M = \{a \in L : a^* = a\}$, defines an involutive $L^*$-automorphism $S$ of $L$ ($S|K = \mathrm{id}$ and $S|M = -\mathrm{id}$) which can be extended to $\tilde{L}$ by linearity. $S$ is called the involution of $\tilde{L}$ associated to $L$. Conversely, if $S$ is an involutive $L^*$-automorphism of $\tilde{L}$, then $S$ leaves the unique compact form $U$ (set of all skew-adjoint elements of $\tilde{L}$) invariant and we have $U = K + iM$, the decomposition of $U$ into eigenspaces of $S$. The real form $L = K + M$ is said to be associated to $S$.

There is a one-to-one correspondence between conjugacy classes of

*AMS* 1970 *subject classifications*. Primary 22E65; Secondary 17B65.

*Key words and phrases*. $L^*$-algebras, Cartan subalgebras, Cartan decompositions, Hilbert-Schmidt operators.

[1] These results are contained in the author's doctoral dissertation written under the direction of Professor Ichiro Satake at the University of California, Berkeley.

real forms and conjugacy classes of $L^*$-automorphisms of $\tilde{L}$ containing an involutive element.

Following an idea of S. Murakami [9], [10], the following can be proved:

THEOREM 2. *Let $\tilde{L}$ be a semisimple complex $L^*$-algebra and $S$ be an involutive $L^*$-automorphism of $\tilde{L}$. Then, there exist a Cartan subalgebra $\tilde{H}$ and a regular selfadjoint element $h$ in it such that: $S\tilde{H}=\tilde{H}$, $Sh=h$ and the* 1*-eigenspace of $S$ in $\tilde{H}$ is a maximal abelian $L^*$-subalgebra of $\tilde{K}$ (the complexification of $K$).*

Having such a Cartan subalgebra it is possible to determine explicitly the structure of $\tilde{K}$ in terms of the roots of $\tilde{L}$ relative to $\tilde{H}$.

THEOREM 3. *Let $\tilde{L}$ be a simple complex $L^*$-algebra, $\tilde{H}$ be a Cartan subalgebra, and $\Delta$ be the root system of $\tilde{L}$ relative to $\tilde{H}$. Then, if an involutive rotation leaves a regular selfadjoint element fixed, it is a "particular rotation" (i.e. it leaves a system of simple roots invariant).*

It is known, [4], that in the case of simple $L^*$-algebras of types A and C all Cartan subalgebras are conjugate, and in case B the Cartan subalgebras fall into two conjugacy classes. Thus, if we fix in cases A and C a Cartan subalgebra $\tilde{H}$ and a system of simple roots $\pi$, there exists in each conjugacy class of $L^*$-automorphisms containing an involutive element, an involution leaving $\tilde{H}$ and $\pi$ invariant. In case B we have to take two nonconjugate Cartan subalgebras in order to get a similar result.

The classification follows easily by reducing such an involution to a normal form.

The result we obtain is exactly what we expect as an infinite-dimensional analogue of classical simple Lie algebras.

**Summary of the results.** Let $E$ be a separable Hilbert space, and $\Phi=\{e_i\}$ be an o.n.b., which we are going to reorder in different ways according to the case under consideration. $\mathfrak{gl}(\infty, \mathbf{C})_2$, the set of all Hilbert-Schmidt operators of $E$, is a simple complex $L^*$-algebra of type A. $\mathfrak{o}(\infty, \mathbf{C})_2=\{a\in\mathfrak{gl}(\infty, \mathbf{C})_2 : {}^t a=-a\}$ is a simple complex $L^*$-algebra of type B. Let $\Phi=\{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$ and $J=\left[\begin{smallmatrix} 0 & I \\ -I & 0\end{smallmatrix}\right]$, i.e. $J$ is the bounded operator of $E$ defined by $Je_{-i}=-e_i$ and $Je_i=e_{-i}$; then $\mathfrak{sp}(\infty, \mathbf{C})_2=\{a\in\mathfrak{gl}(\infty, \mathbf{C})_2 : {}^t aJ+Ja=0\}$ is a simple complex $L^*$-algebra of type C. We note that in this case we can turn $E$ into a right vector space over $\mathbf{K}$ ($\mathbf{K}=\{l, i, j, ij\}_{\mathbf{R}}$ the algebra of quaternions) by defining the action of $j$ by $xj=J\bar{x}$ for all $x\in E$; an o.n.b. of $E$ over $\mathbf{K}$ is $\{e_1, e_2, \cdots\}$. An element $a\in\mathfrak{gl}(\infty, \mathbf{C})_2$ is $\mathbf{K}$-linear if and only if

$J\bar{a}=aJ$, i.e. if $a$ is of the form $\left[\begin{smallmatrix} a_1 & a_2 \\ -\bar{a}_2 & \bar{a}_1 \end{smallmatrix}\right]$, and when this is so, we shall use the matrix expression of $a$ given by $a_1+a_2j$, in other words, as a linear operator of $E$ over $K$. We denote by $\mathfrak{gl}(\infty, K)_2$ the set of all $K$-linear operators in $\mathfrak{gl}(\infty, C)_2$.

The *simple separable real $L^*$-algebras having a complex structure* are the real $L^*$-algebras obtained from $\mathfrak{gl}(\infty, C)_2$, $\mathfrak{o}(\infty, C)_2$ and $\mathfrak{sp}(\infty, C)_2$ by restriction of scalars.

The *compact simple separable real $L^*$-algebras* are

$$\mathfrak{u}(\infty, C)_2 = \{a \in \mathfrak{gl}(\infty, C)_2 : a^* = -a\},$$
$$\mathfrak{o}(\infty, R)_2 = \{a \in \mathfrak{o}(\infty, C)_2 : a^* = -a\},$$
$$\mathfrak{u}(\infty, K)_2 = \{a \in \mathfrak{gl}(\infty, K)_2 : {}^t\bar{a} + a = 0\},$$

where $\bar{x}=x_0-x_1i-x_2j-x_3ij$ if $x=x_0+x_1i+x_2j+x_3ij$ in $K$.

In the following, $\tilde{L}$ will denote a simple complex $L^*$-algebra, $S$ an involutive $L^*$-automorphism of $\tilde{L}$, and $L$ the real form of $\tilde{L}$ associated to $S$ or a real form of $\tilde{L}$ conjugate to $L$.

The *noncompact simple separable real $L^*$-algebras* are

(a) $\Phi = \{e_1, e_2, \cdots, e_n, \cdots\}$, $K_n = \begin{bmatrix} -I_n & 0 \\ 0 & I \end{bmatrix}$,

AI
$$\tilde{L} = \mathfrak{gl}(\infty, C)_2, \quad Sa = -{}^ta,$$
$$L = \mathfrak{gl}(\infty, R)_2 = \text{all real matrices in } \mathfrak{gl}(\infty, C)_2.$$

AIII($n$)
$$\tilde{L} = \mathfrak{gl}(\infty, C)_2, \quad Sa = K_naK_n^{-1},$$
$$L = \mathfrak{u}(n, \infty)_2 = \{a \in \mathfrak{gl}(\infty, C)_2 : {}^t\bar{a}K_n + K_na = 0\}.$$

BDI($n$)
$$\tilde{L} = \mathfrak{o}(\infty, C)_2, \quad Sa = K_naK_n^{-1},$$
$$L = \mathfrak{o}(n, \infty)_2 = \{a \in \mathfrak{gl}(\infty, R)_2 : {}^taK_n + K_na = 0\}.$$

(b) $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$, $K_\infty = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$,

AIII($\infty$)
$$\tilde{L} = \mathfrak{gl}(\infty, C)_2, \quad Sa = K_\infty aK_\infty,$$
$$L = \mathfrak{u}(\infty, \infty)_2.$$

BDI($\infty$)
$$\tilde{L} = \mathfrak{o}(\infty, C)_2, \quad Sa = K_\infty aK_\infty^{-1},$$
$$L = \mathfrak{o}(\infty, \infty)_2.$$

(c) $\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}$, $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$,

AII
$$\tilde{L} = \mathfrak{gl}(\infty, C)_2, \quad Sa = -J{}^taJ^{-1},$$
$$\tilde{L} = \mathfrak{gl}(\infty, K)_2.$$

**CI**
$$L = \mathfrak{sp}(\infty, C)_2, \qquad Sa = \bar{a},$$
$$L = \mathfrak{sp}(\infty, R)_2 = \text{all real matrices in } \mathfrak{sp}(\infty, C)_2.$$

(d)
$$\Phi = \{e_{-1}, e_{-2}, \cdots, e_1, e_2, \cdots\}, \qquad K_{n,n} = \begin{bmatrix} K_n & 0 \\ 0 & K_n \end{bmatrix},$$

**CII($n$)**
$$\tilde{L} = \mathfrak{sp}(\infty, C)_2, \qquad Sa = K_{n,n} a K_{n,n}^{-1},$$
$$L = \mathfrak{u}(n, \infty, K) = \{a \in \mathfrak{gl}(\infty, K) : {}^t\bar{a} K_n + K_n a = 0\},$$

here $K_n$ is the operator of $E$ over $K$ defined by $K_n e_i = -e_i$ $(1 \leqq i \leqq n)$ and $K_n e_i = e_i$ $(i > n)$.

(e)
$$\Phi = \{e_{-1}, e_{-3}, \cdots, e_{-2}, e_{-4}, \cdots, e_1, e_3, \cdots, e_2, e_4, \cdots\},$$
$$K_{\infty,\infty} = \begin{bmatrix} K_\infty & 0 \\ 0 & K_\infty \end{bmatrix},$$

**DIII**
$$\tilde{L} = \mathfrak{o}(\infty, C)_2, \qquad Sa = JaJ^{-1},$$
$$L = \mathfrak{o}(\infty, K)_2 = \{a \in \mathfrak{gl}(\infty, K)_2 : {}^t\bar{a} + a = 0\},$$

here $\bar{x} = x_0 + x_1 i - x_2 j + x_3 ij$ if $x = x_0 + x_1 i + x_2 j + x_3 ij$ in $K$.

**CII($\infty$)**
$$\tilde{L} = \mathfrak{sp}(\infty, C)_2, \qquad Sa = K_{\infty,\infty} a K_{\infty,\infty}^{-1},$$
$$L = \mathfrak{u}(\infty, \infty, K)_2 \qquad (\text{see CII}(n)).$$

As a result of the above considerations, we obtain the following:

THEOREM 4. *Two real forms $L$ and $L'$ of a simple complex $L^*$-algebra are $L^*$-isomorphic if and only if the corresponding characteristic subalgebras are $L^*$-isomorphic.*

## REFERENCES


1. V. K Balachandran, *The Weyl group of an $L^*$-algebra*, Math. Ann. **154** (1964), 157–165. MR **28** #4374.

2. ———, *Regular elements of $L^*$-algebras*, Math. Z. **93** (1966), 161–163. MR **33** #2689.

3. ———, *Simple systems of roots in $L^*$-algebras*, Trans. Amer. Math. Soc. **130** (1968), 513–524. MR **38** #199.

4. ———, *Simple $L^*$-algebras of classical type*, Math. Ann. **180** (1969), 205–219. MR **39** #4684.

5. V. K. Balachandran and P. R. Parthasarathy, *Cartan subalgebras of an $L^*$-algebra*, Math. Ann. **166** (1966), 300–301. MR **34** #3348.

6. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. **7**, Interscience, New York, 1958. MR **22** #8302.

7. S. Helgason, *Differential geometry and symmetric spaces*, Pure and Appl. Math., vol. 12, Academic Press, New York, 1962. MR **26** #2986.

**8.** N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. **10**, Interscience, New York, 1962. MR **26** #1345.

**9.** S. Murakami, *On the automorphisms of a real semi-simple Lie algebra*, J. Math. Soc. Japan **4** (1952), 103–133. MR **14**, 531.

**10.** ———, *Sur la classification des algèbres de Lie réelles et simples*, Osaka University, Osaka, Japan (mimeographed notes).

**11.** J. R. Schue, *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. **95** (1960), 69–80. MR **22** #8352.

**12.** ———, *Cartan decompositions for $L^*$-algebras*, Trans. Amer. Math. Soc. **98** (1961), 334–349. MR **24** #3242.

**13.** F. Gantmacher, *Canonical representation of automorphisms of a complex semi-simple Lie group*, Mat. Sb. **5** (47) (1939), 101–146. MR **1**, 163.

**14.** ———, *On the classification of real simple Lie groups*, Mat. Sb. **5** (47) (1939), 217–249. MR **2**, 5.

University of Maryland, College Park, Maryland 20742