BIHARMONIC CLASSIFICATION OF
RIEMANNIAN MANIFOLDS[1]

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We consider a noncompact orientable C^∞ manifold R of dimension $m \geq 2$ with a C^∞ Riemannian metric $ds^2 = \sum_{i,j=1}^{m} g_{ij}\, dx^i\, dx^j$. The corresponding Laplace-Beltrami operator is

$$(1) \qquad \Delta \cdot = -g^{-1/2} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{m} g^{1/2} g^{ij} \frac{\partial \cdot}{\partial x^j} \right),$$

with $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Let φ and ψ be continuous functions on the Wiener or Royden boundary β of R (cf. [2] and [5]; the terminology used in the present note is adopted from [5]). We are interested in solving the biharmonic boundary value problem

$$(2) \qquad \Delta^2 u = 0, \qquad u \,|\, \beta = \varphi, \qquad \Delta u \,|\, \beta = \psi.$$

Here u and Δu are required to be continuously extendable to the Wiener or Royden compactification R^* of R. If $\varphi = \psi = 0$ on the harmonic boundary $\delta \subset \beta$, then the solution of (2) vanishes identically. The problem, therefore, is to solve

$$(3) \qquad \Delta^2 u = 0, \qquad u \,|\, \delta = \varphi, \qquad \Delta u \,|\, \delta = \psi.$$

Such problems are of fundamental importance in applications of biharmonic functions to physics, in particular the theory of bending of thin plates (cf. [1]). Unconditional solvability cannot be expected, since the solution of (3) clearly exists for the unit ball R in the Euclidean space E^m but not for $R = E^m$. This simple example illustrates the significance of the biharmonic classification problem in the theory of elasticity. Here we shall announce some results on the problem; the proofs will appear elsewhere (cf. [3], [4]).

1. Quasiharmonic functions.
The simplest nonharmonic biharmonic functions are what we shall call *quasiharmonic functions* u, characterized by $\Delta u = \text{const} \neq 0$. The normalized class $Q = Q(R) = \{u \in C^2(R) \,|\, \Delta u = 1\}$ can be viewed as a special class of super-

harmonic functions on R. Let P, B, and D stand for the classes of positive, bounded, and Dirichlet finite functions, and set $BD = B \cap D$. Consider the class O_{QX} of Riemannian ma iifolds R on which $QX(R) = \varnothing$. Denote by $G(x, y)$ the harmonic Green's function on $R \notin O_G$ and set $G(x, y) = \infty$ on $R \in O_G$. We have the following complete characterizations:

$$R \in O_{QP} \quad \text{if and only if } a = \int_R G(x, y)\, dy = \infty\,;$$

$$R \in O_{QB} \quad \text{if and only if } b = \sup_{x \in R} \int_R G(x, y)\, dy = \infty\,;$$

(4)

$$R \in O_{QD} \quad \text{if and only if } c = \int_{R \times R} G(x, y)\, dx\, dy = \infty\,;$$

$$R \in O_{QBD} \text{ if and only if } b = \infty \text{ or } c = \infty.$$

Here $dx = g^{1/2}\, dx^1 \cdots dx^m$ is the Riemannian volume element on R.

By means of these characterizations we also obtain the inclusion relations

(5)
$$O_G \to O_{QP} \begin{array}{c} \nearrow O_{QB} \searrow \\ \vdots \\ \searrow O_{QD} \nearrow \end{array} O_{QBD}.$$

Here $X \to Y$ stands for $X \underset{\neq}{\subsetneq} Y$, and $X \frown Y$ means that $X \not\subset Y$ and $Y \not\subset X$.

2. **Riesz representation.** For a class X of functions and an operation T on certain functions we denote by X_T the class $\{f \mid Tf \text{ is defined and } Tf \in X\}$. Let $W = W(R)$ be the class of *biharmonic functions* u on R, defined by $\Delta^2 u = 0$ with $u \in C^4(R)$. Suppose $u \in WXP_\Delta(R)$ with $X = P$, B, or D. We can establish the Riesz representation

(6)
$$u(x) = H_1 u(x) + \int_R G(x, y) H_2 u(y)\, dy,$$

with $H_1 u \in HX(R)$ and $H_2 u = \Delta u \in HP(R)$. The decomposition is unique. We call the function $H_1 u$ the *harmonic part* of u, and the function $GH_2 u$ the *potential part* of u, with

(7)
$$(Gf)(x) = \int_R G(x, y) f(y)\, dy.$$

Here u, $H_i u$ $(i = 1, 2)$, and $GH_2 u$ are continuous on the Wiener com-

pactification $R^* = R_N^*$ of R, and $u = H_1 u$ or equivalently $GH_2 u = 0$ on the Wiener harmonic boundary $\delta = \delta_N$. In the case $X = D$, we can replace R_N^* and δ_N by the Royden compactification R_M^* and the Royden harmonic boundary δ_M, except regarding $H_2 u$, and obtain

$$(8) \qquad D_R(u) = D_R(H_1 u) + \int_{R \times R} G(x, y) H_1 u(x) H_2 u(y) \, dx \, dy.$$

The mapping $\gamma u = (H_1 u, H_2 u)$ is an injective linear transformation from WXP_Δ into $HX \times HP$. To describe the image of γ, let $F = \{f | f \text{ is measurable}\}$ and $F_0 = \{f \in F | G|f|(x) < \infty\}$. Then clearly

$$(9) \qquad\qquad \gamma(WXP_\Delta) = HX \times HPF_0.$$

3. **Classes of biharmonic functions.** For the sake of brevity we also use the notation C for BD whenever it is obvious that we are not referring to the class of continuous functions. The solvability of (3) depends on the existence of the representation (6) and the surjectivity of γ for the class WBB_Δ and its subclasses WBC_Δ, WCB_Δ, and WCC_Δ. In view of

$$(10) \qquad\qquad O_G \subsetneqq O_{WBB_\Delta},$$

we may suppose that $M \notin O_G$. As a consequence of (6) we conclude:
  If $R \notin O_{QP}$, then the representation (6) is valid for $u \in WBB_\Delta$.
We stress that in (6) for $u \in WBB_\Delta$ we have the requirement

$$\int_R G(x, y) \, | H_2 u(y) | \, dy < \infty.$$

The unrestricted solvability of (3) is then equivalent to the surjectivity of γ, which is characterized as follows:

$$(11) \quad \begin{aligned} \gamma(WBB_\Delta) &= HB \times HB \quad \text{if and only if } R \notin O_{QB}; \\ \gamma(WBC_\Delta) &= HB \times HC \quad \text{if and only if } R \notin O_{QB}; \\ \gamma(WCB_\Delta) &= HC \times HB \quad \text{if and only if } R \notin O_{QC}; \\ \gamma(WCC_\Delta) &= HC \times HC \quad \text{if and only if } R \notin O_{QC}. \end{aligned}$$

4. **Biharmonic projection.** The boundary functions in (3) are restrictions to δ of functions in the Wiener algebra N or the Royden algebra M on R. The following decomposition suggests another approach to solving (3). Let $F_1 = \{f \in F_0 | \sup_{x \in M} | Gf(x) | < \infty\}$ and $F_2 = \{f \in F | G(|f|, |f|) < \infty\}$. Here

$$G(f, g) = \int_{R \times R} G(x, y) f(x) g(y) \, dx \, dy,$$

which already occurred in (8), is referred to as the *mutual energy* of f and g. If $f_i \in F_2$ and $Gf_i \in C^2(R)$ $(i=1, 2)$, then

$$(12) \qquad\qquad D_R(Gf_1, Gf_2) = G(f_1, f_2).$$

Let

$$\Phi_1 = NN_\Delta(F_1)_\Delta(F_1)_{\pi\Delta}, \qquad \Phi_2 = NM_\Delta(F_1)_\Delta(F_1)_{\pi\Delta},$$

$$\Phi_3 = MN_\Delta(F_1)_\Delta(F_1)_{\pi\Delta}(F_2)_\Delta(F_2)_{\pi\Delta},$$

$$\Phi_4 = MM_\Delta(F_1)_\Delta(F_1)_{\pi\Delta}(F_2)_\Delta(F_2)_{\pi\Delta},$$

where π is the harmonic projection on N or M. We denote by N_δ and M_δ the potential subalgebras of N and M, defined by $N_\delta | \delta = 0$ and $M_\delta | \delta = 0$. The following direct decompositions are valid:

$$(13) \quad \begin{aligned} \Phi_1 &= W\Phi_1 + (N_\delta\Phi_1)(N_\delta)_\Delta; & \Phi_2 &= W\Phi_2 + (N_\delta\Phi_2)(M_\delta)_\Delta; \\ \Phi_3 &= W\Phi_3 + (M_\delta\Phi_3)(N_\delta)_\Delta; & \Phi_4 &= W\Phi_4 + (M_\delta\Phi_4)(M_\delta)_\Delta. \end{aligned}$$

We shall call the projection $\tau_i f$ of $f \in \Phi_i$ to $W\Phi_i$ the *biharmonic projection*. It is given by

$$(14) \qquad \begin{aligned} \tau_i f &= \pi f + G\pi\Delta f, \\ f - \tau_i f &= G(\Delta f - \pi\Delta f). \end{aligned}$$

Here $W\Phi_1 \subset WBB_\Delta$, $W\Phi_2 \subset WBC_\Delta$, $W\Phi_3 \subset WCB_\Delta$, and $W\Phi_4 \subset WCC_\Delta$. The inclusions are replaced by equalities if $R \notin O_{QB}$, $R \notin O_{QB}$, $R \notin O_{QC}$, and $R \notin O_{QC}$, respectively; in these cases $\Phi_1 = NN_\Delta$, $\Phi_2 = NM_\Delta$, $\Phi_3 = MN_\Delta$, and $\Phi_4 = MM_\Delta$.

5. **Biharmonic measure.** Let $P(x, \zeta)$ with $(x, \zeta) \in R \times \beta$ be the harmonic kernel. We shall refer to the function

$$(15) \qquad\qquad K(x, \zeta) = \int_R G(x, y)P(y, \zeta)\, dy,$$

which may or may not be finite, as the *biharmonic kernel*. If it is finite, then

$$(16) \qquad\qquad \Delta_x K(x, \zeta) = P(x, \zeta).$$

Suppose $\varphi \in HB | \delta = N | \delta$ and $\psi \in HBF_1 | \delta = NF_1 | \delta$. Then the solution of (3) is given by

$$(17) \qquad u(x) = \int_\delta (P(x, \zeta)\varphi(\zeta) + K(x, \zeta)\psi(\zeta))\, d\mu(\zeta),$$

where μ is the harmonic measure on δ. If φ and ψ are μ-measurable on

δ and the integral in (17) is well defined, then the function (17) is biharmonic on R. If, moreover, φ (resp. ψ) is continuous at some $\zeta \in \delta$, then u (resp. Δu) has the continuous boundary value $\varphi(\zeta)$ (resp. $\psi(\zeta)$) at ζ.

We define the *biharmonic measure* of a Borel set $E \subset \beta$ relative to $x \in R$ as

$$(18) \qquad \kappa_x(E) = \int_E K(x, \zeta) \, d\mu(\zeta),$$

and the *absolute biharmonic measure* as

$$(19) \qquad \kappa(E) = \sup_{x \in M} \kappa_x(E).$$

The latter is not a genuine measure but an outer measure on β. We call a point $\zeta \in \beta$ *absolutely singular* if $\kappa(U) = \infty$ for every neighborhood U of ζ in β. The set S of such points constitutes a compact subset of δ. We consider the function

$$l_E(x) = \int_E P(x, \zeta) \, d\mu(\zeta)$$

for a Borel set $E \subset \delta - S$. By substituting the class

$$Q_S = \{ u \in C^2(R) \mid \Delta u = l_E, \, E \subset \delta - S \}$$

for Q in §1, we can deduce similar but more precise results than those in §§3 and 4. An analysis of the classes $O_{Q_S X}$ $(X = P, B, D, C)$ and, from a somewhat more general viewpoint, the classification problem for O_{WXY_Δ} $(X, Y = B, C)$ are important further topics; here O_{WXY_Δ} is the class of Riemannian manifolds R on which $WXY_\Delta(R)$ reduces to constants.

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