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ORIENTATION-PRESERVING MAPPINGS, A SEMIGROUP OF GEOMETRIC TRANSFORMATIONS AND A CLASS OF INTEGRAL OPERATORS[1]

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Let A, B be smooth $(=C^\infty)$, oriented n-manifolds, A with naturally oriented boundary, ∂A, and B without boundary.

A very important problem in geometric analysis is that of giving an algebraic and/or combinatorial characterization of those smooth mappings from ∂A to B which can be extended to a smooth, orientation-preserving mapping from A to B.

In this work, one such characterization is given in the particular case where A is the unit disk, D $(\partial D = S^1)$, and B is the plane, R^2. An application is made to a class of convolution-type operators to show they are topologically equivalent to the Hilbert transform.

1. **Preliminaries.** A smooth $f: S^1 \to R^2$ is called *extendable* if there is a smooth $F: D^- \to R^2$ (D^- closure of D) with nonnegative Jacobian, J_F, and whose restriction to S^1 is f. If, further, $J_F > 0$ on S^1 then f is *properly extendable.*

A *Titus transformation* T is a linear operator on the vector space of smooth functions from S^1 to R^2 given by:

$$(Tf)(t) = f(t) + c(t) \det[v, f'(t)]v, \tag{1.1}$$

c a nonnegative, smooth function on S^1. The set of all finite compositions of Titus transformations is a semigroup, $\mathfrak{T}$. The effect of a Titus transformation can be represented by an elementary operation of growth along a fixed direction, growth understood in the sense of moving to the outside of an oriented curve.

A "degenerate" mapping $f: S^1 \to R^2$ is one whose image lies in a one-dimensional subspace. A *Titus mapping* (T-mapping) is the image by an element of $\mathfrak{T}$ of a degenerate mapping. A Titus mapping, thus, has

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a simple, basic, geometric meaning: it is a mapping which can be obtained as a finite number of growth operations applied to a degenerate curve.

2. Results.

THEOREM 1. *A normal mapping* (*see* [4] *for definition*) *is extendable if and only if it is a Titus mapping.*

The *if* part is proved by induction and is due to C. J. Titus (oral communication). The *only if* part follows from Theorem 2 below.

THEOREM 2. *Every properly extendable mapping is a Titus mapping.*

The proof of Theorem 2 is accomplished by reduction to the case of a holomorphic function having only a simple zero of the derivative, in which case a direct construction is performed.

3. Application.

Consider integral operators given by:

$$y(t) = -\int_0^{2\pi} k(s)x(t-s)\,ds, \tag{2.1}$$

where x is smooth and has period 2π, and k is real analytic in $(0, 2\pi)$ with

$$k'(s) = \int_{-\infty}^{\infty} e^{-sr}\,d\mu(r), \tag{2.2}$$

μ a nondecreasing function (see [1], [2]). We allow certain cases where k is not integrable; the integral in (2.1) must, then, be interpreted in the sense of Cauchy's principal value. (See [2] for details.)

Such an operator will be called a *BL-operator*. A mapping $f: S^1 \to R^2$ is called a *BL-mapping* if it can be expressed as $f(t) = x(t) + iy(t)$ where x, y satisfy (2.1) and (2.2).

It is easy to see that a holomorphic boundary [3] is a BL-mapping. We prove that they are generic, in some sense.

THEOREM 3. *Every normal BL-mapping is topologically equivalent to a holomorphic boundary.*

The proof uses approximation by T-mappings and results of Stoïlow-Whyburn-Carathéodory [3] about extendable mappings.

Full details will be published elsewhere.

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