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C^*-ALGEBRAS GENERATED BY MEASURES[1]

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We announce here some results dealing with nonabelian extensions of the theory of almost periodic functions to the duals of compact groups. For G a locally compact group, let $\hat{G}$ be the dual of G (the set of equivalence classes of continuous, irreducible, unitary representations of G). For $\pi \in \hat{G}$ and $\mu \in M(G)$, the measure algebra of G, let $\pi(\mu)$ be the Fourier-Stieltjes transform of μ at π. Let $\|\mu\|_\infty$ be $\sup\{\|\pi(\mu)\| : \pi \in \hat{G}\}$, and let $\mathfrak{M}(\hat{G})$ be the C^*-completion of $M(G)$ relative to the norm $\|\cdot\|_\infty$. Let $\mathfrak{M}_a(\hat{G})$, $\mathfrak{M}_d(\hat{G})$ be the closures in $\mathfrak{M}(\hat{G})$ of $L^1(G)$ (the space of measures absolutely continuous with respect to left Haar measure), $M_d(G)$ (the space of discrete measures) respectively. The algebra $\mathfrak{M}_d(\hat{G})$ is a nonabelian analogue of the classical algebra of almost periodic functions. A standard reference for C^*-algebras is [1].

We denote the spectrum of $\mathfrak{M}(\hat{G})$ by $\kappa\hat{G}$. In the abelian case this is the closure of the dual group of G in the spectrum of $M(G)$. In general $\hat{G}$ is identified with a dense open subset of $\kappa\hat{G}$ and $\kappa\hat{G}\backslash\hat{G}$ is the annihilator of $\mathfrak{M}_a(\hat{G})$. We investigate the C^*-extension of the canonical projection which maps a measure to its discrete part. This makes possible a proof that $\kappa\hat{G}\backslash\hat{G}$ contains a homeomorphic copy of the reduced dual of G_d, the group G made discrete. We further show that if G is nondiscrete and G_d is amenable then the sup and lim sup norms are identical on $\mathfrak{M}_d(\hat{G})$, and if $\mu \in \mathfrak{M}_d(\hat{G})$ then $\mu \in M_d(\hat{G})$ ($\mu \in M(G)$).

For $S \subset \kappa\hat{G}$ let $\mathfrak{N}(S) = \{\phi \in \mathfrak{M}(\hat{G}) : \pi(\phi) = 0 \text{ for all } \pi \in S\}$. Then $\mathfrak{N}(S)$ is a closed ideal in $\mathfrak{M}(\hat{G})$. Let $\mathfrak{M}(S) = \mathfrak{M}(\hat{G})/\mathfrak{N}(S)$ be the quotient C^*-algebra.

Denote the locally compact group G made discrete by G_d. Then $\hat{G}_d$ is the dual of G_d and is also the spectrum of $\mathfrak{M}(\hat{G}_d) = \mathfrak{M}_a(\hat{G}_d) = \mathfrak{M}_d(\hat{G}_d)$. Each $\pi \in \hat{G}$ gives an irreducible unitary representation of G_d; thus $\hat{G}$ is identified with a subset of $\hat{G}_d$. We denote the closure of $\hat{G}$ in $\hat{G}_d$ by $\hat{G}_{dc}$. Further denote the reduced dual of G_d by $\hat{G}_{dr}$, the set of $\pi \in \hat{G}_d$ which are weakly contained in the left regular representa-

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tion of G_d on $l^2(G_d)$. Observe that $M_d(G)$ can be identified with $M(G_d)$, and $\mathfrak{M}_d(\hat{G}) \cong \mathfrak{M}(\hat{G}_{dc})$.

THEOREM 1. *There is a unique C^*-homomorphism of $\mathfrak{M}(\hat{G})$ onto $\mathfrak{M}(\hat{G}_{dr})$ such that for $\mu \in M(G)$, $P\mu$ is the discrete part of μ, and* kernel $P \supset \mathfrak{N}(\hat{G}_r)$.

COROLLARY 2. *For $\mu \in M_d(G)$, $\|\mu\|_{dr} \leqq \|\mu\|_r \leqq \|\mu\|_\infty$; and thus $\hat{G}_{dc} \supset \hat{G}_{dr}$.*

COROLLARY 3. *If G is nondiscrete and $\pi \in \hat{G}_{dr}$ then $\pi \circ P$ is an irreducible representation of $\mathfrak{M}(\hat{G})$ and $\pi \circ P \in \kappa\hat{G} \backslash \hat{G}$. Further the map $\pi \rightarrow \pi \circ P$ is a homeomorphism of $\hat{G}_{dr}$ into $\kappa\hat{G} \backslash \hat{G}$.*

Let G be nondiscrete and $S \subset \hat{G}$. Then define a seminorm on $\mathfrak{M}(\hat{G})$, called S-lim sup, to be the quotient norm of $\mathfrak{M}(S)/\mathfrak{M}_a(S)$. Recall $\mathfrak{M}(S) = \mathfrak{M}(\hat{G})/\mathfrak{N}(S)$ and $\mathfrak{M}_a(S) = \mathfrak{M}_a(\hat{G})/(\mathfrak{N}(S) \cap \mathfrak{M}_a(\hat{G}))$. If G is compact or abelian then the $\hat{G}$-lim sup is identical to $\limsup_{\pi \to \infty} |\pi(\phi)| = \inf_K \{\sup |\pi(\phi)|, \pi \notin K\}$, K a compact subset of $\hat{G}$, for $\phi \in \mathfrak{M}(\hat{G})$.

A locally compact group G is said to be amenable if there exists a left invariant mean on the space of bounded continuous functions. Equivalent characterizations are that $\hat{G} = \hat{G}_r$, or that the representation $G \rightarrow \{1\}$ is in $\hat{G}_r$.

Under the assumption that G_d is amenable, we can prove direct extensions of certain abelian-case theorems.

THEOREM 4. *If G_d is amenable, $\phi \in \mathfrak{M}_d(\hat{G})$, then $\hat{G}$-lim sup$(\phi) = \|\phi\|_\infty$. Further if $\mu \in M(G)$, then $\|\mu\|_\infty \geqq \hat{G}$-lim sup$(\mu) \geqq \|P\mu\|_\infty = G$-lim sup$(P\mu)$.*

THEOREM 5. *If G_d is amenable, then $\mathfrak{M}(\hat{G}) = \mathfrak{M}_c(\hat{G}) \oplus \mathfrak{M}_d(\hat{G})$, where $\mathfrak{M}_c(\hat{G})$ is the closure in $\mathfrak{M}(\hat{G})$ of the set of continuous measures in $M(G)$.*

COROLLARY 6. *If G_d is amenable and $\mu \in M(G)$ and $\mu \in \mathfrak{M}_d(\hat{G})$ then $\mu \in M_d(G)$.*

If G_d is amenable, then Corollary 2 reduces to: for $\mu \in M_d(G)$, $\|\mu\|_{dr} = \|\mu\|_r = \|\mu\|_\infty$. This fact has been shown by Zeller-Meier in [2].

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