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A RELATION BETWEEN TWO SIMPLICIAL ALGEBRAIC K-THEORIES

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There is a proliferation of proposed algebraic K-theories [5], [6], [8], [11], [12], [13], [15] and one of the present authors can share the blame for three of them. However some rather striking relationships have been found which indicate that the various K-theories, while not the same, are at any rate comparable. This note describes a relation between the K-theory proposed by Quillen [13], which has the advantage of computability using powerful techniques of the homology of groups, and that K-theory defined axiomatically in [8] and constructed semisimplicially in [5], which possesses extremely pleasant functorial properties. It is our hope that this connection will be useful in computing the K-theory of [8], and thus eventually the stable K-theory [7] which is analogous, in this rarefied setting of rings, with stable homotopy theory.

We begin by recalling (in slightly different form from [13]) Quillen's construction. For any ring R, one forms $\mathbf{Z}_\infty \overline{W}(\mathrm{Gl}(R))$. Here $\mathrm{Gl}(R)$ is regarded as a (constant) simplicial group, $\overline{W}$ is the simplicial classifying space, [10, p. 87], and $\mathbf{Z}_\infty$ is the integral completion functor of Bousfield and Kan [2]. Then $K_i^Q(R) = \pi_i(\mathbf{Z}_\infty \overline{W}\mathrm{Gl}(R))$, $i \geqq 1$, where the superscript refers to the author.

In order to give the simplicial definition of [5] of the K-theory of [8], we recall some terminology. One works in the category *ring* of rings (without unit) and one lets E be the endomorphism of *ring*, $ER = tR[t]$, the *path ring*. The morphisms $\epsilon: E \to I$, $\mu \to E^2$ given by

$$\epsilon_R: ER = tR[t] \to R, \qquad \text{“}t \to 1\text{,”} \quad \text{and}$$

$$\mu_R: ER = tR[t] \to tuR[t, u] = E^2R, \qquad t \to tu,$$

give rise to the cotriple (E, ϵ, μ) in *ring*. Let $\overline{E}R$ be the augmented semisimplicial complex, $(\overline{E}R)_n = E^{n+2}R$, $n \geqq -1$, constructed from this cotriple, and set

$$K^{-i}(R) = \tilde{\pi}_{i-2}(\mathrm{Gl}(\overline{E}R)), \qquad i \geqq 1.$$

The upper indexing is motivated by topological considerations, and

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$\tilde{\pi}$ refers to the "augmented" homotopy groups [5] ($\tilde{\pi}_i = \pi_i$ for $i \geqq 1$, the augmentation entering for $i < 1$ because of the extra face operator $\epsilon: (\overline{E}R)_0 \to (\overline{E}R)_{-1}$).

Consider now the cotriple (P, ϵ, μ) in *ring* where $PR = R[t]$, $\epsilon_R: PR \to R$ is given by "$t \to 1$" and $\mu_R: PR = R[t] \to P^2R = R[t, u]$ is given by $t \to tu$. Let $\overline{P}R$ be the associated augmented semisimplicial complex. Then there is a canonical map

$$\iota: \mathrm{Gl}(\overline{E}R) \to \mathrm{Gl}(\overline{P}R).$$

Also, the complex $\mathrm{Gl}(\overline{P}R)$ is acyclic, so if H is the homogeneous space of the inclusion ι, then $H \simeq \overline{W}\mathrm{Gl}(\overline{E}R)$. We can however identify H explicitly.

Note that $(\overline{P}R)_n = P^{n+1}R = R[t_0, \cdots, t_n]$ and $(\overline{E}R)_n = (t_0 \cdots t_n) \cdot R[t_0, \cdots, t_n]$. Thus we have a short exact sequence of rings

$$(\overline{E}R)_n \to (\overline{P}R)_n \to Q(R)_n,$$

where $Q(R)_n = R[t_0, \cdots, t_n]/(t_0 \cdots t_n)$, $n \geqq 0$. $Q(R)$ is a simplicial ring, and since Gl is left exact, we have an exact sequence of simplicial groups

$$1 \to \mathrm{Gl}(\overline{E}R) \xrightarrow{\iota} \mathrm{Gl}(\overline{P}R) \xrightarrow{j} Q(R).$$

THEOREM 1. *The canonical map $H \to QR$ is an isomorphism of simplicial groups. In particular, the map j above is surjective.*

Note that $Q(R)_0 = \mathrm{Gl}(R[t_0]/(t_0)) = \mathrm{Gl}(R)$. Thus we have an imbedding of the constant complex $\mathrm{Gl}(R) \xrightarrow{\alpha} Q(R)$ and hence a map

$$\mathbf{Z}_\infty \overline{W}(\alpha): \mathbf{Z}_\infty \overline{W}\,\mathrm{Gl}(R) \to \mathbf{Z}_\infty \overline{W} Q(R).$$

Note that by Theorem 1, $\mathrm{Gl}(\overline{E}R)$ is the "second loop group" of $\overline{W}Q(R)$, so we can identify $\pi_i \overline{W}Q(R) = K^{-i}(R)$, $i \geqq 1$. In order to proceed further we need

THEOREM 2. *The action of $\pi_i \overline{W}Q(R)$ on $\pi_n \overline{W}Q(R)$ is trivial. In particular, $\overline{W}Q(R)$ is "nilpotent" in the terminology of Bousfield and Kan.*

This is proved by translating the problem to showing that the action of $\mathrm{Gl}(R)$ on the augmented homotopy groups of $\mathrm{Gl}(\overline{E}R)$ is trivial. This in turn is a generalization of the classical Whitehead lemma, which implies the statement of Theorem 2 for $\tilde{\pi}_{-1}$.

COROLLARY. [2]. *For all i we have*

$$K^{-i}(R) \cong \pi_i(\overline{W}QR) \cong \pi_i(\mathbf{Z}_\infty \overline{W}QR),$$

where the last isomorphism is induced by the canonical map $\overline{W}QR \to Z_\infty \overline{W}QR$.

COROLLARY. *The map*

$$Z_\infty \overline{W}(\alpha): Z_\infty \overline{W}\,\mathrm{Gl}(R) \to Z_\infty \overline{W}QR$$

induces natural homomorphisms $\alpha_i: K_i^Q(R) \to K^{-i}(R)$ *for all* $i \geqq 1$.

In low dimensions it is possible to identify α_i. Namely, α_1 is always surjective and corresponds to "reduction modulo unipotents." If R is (left) regular, then α_1 is an isomorphism by a result of Bass, Heller, and Swan [1] and α_2 is surjective by [5, Theorem 6.1]. If k is a finite field and $R = k(t)$, then α_2 is known to be an isomorphism [4]. Also, if R is the rationals, one knows that α_2 is an isomorphism [9].

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