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FUNCTION ALGEBRAS AND THE DE RHAM THEOREM IN PL

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0. **Introduction.** There is a classical contravariant functor on the category of smooth manifolds M which assigns to each M the algebra A of all smooth functions on M, and one uses this functor implicitly throughout differential topology. For example, the de Rham theorem extends the customary derivation $d:A \rightarrow \mathcal{E}(A)$ to a cochain complex $(\Lambda\mathcal{E}(A), d)$ whose homology is isomorphic to the real cohomology of M itself. In this paper we construct a corresponding contravariant functor on the category of piecewise linear manifolds M, which assigns to each M an algebra A of functions on M. We then define a derivation $d:A \rightarrow \mathcal{E}(A)$ and extend it to a cochain complex $(\Lambda\mathcal{E}(A), d)$ whose homology is isomorphic to the real cohomology of M; this is the de Rham theorem in PL. As an application we construct connections and curvature homomorphisms in terms of $(\Lambda\mathcal{E}(A), d)$, to which we apply a real version of the Chern-Weil theorem to compute real Pontrjagin classes of PL manifolds without using the Hirzebruch L-polynomials.

1. **Smoothing homeomorphisms.** A *simplicial decomposition of* $\boldsymbol{R}^n$ *at* 0 is any finite triangulation of $\boldsymbol{R}^n$ into open simplexes such that $0 \in \boldsymbol{R}^n$ is the only 0-simplex. If α and β are any two such simplicial decompositions then we write $\alpha < \beta$ whenever β is a subdivision of α. For any α and β there is a simplicial decomposition γ with $\alpha < \gamma$ and $\beta < \gamma$, so that the simplicial decompositions of $\boldsymbol{R}^n$ at 0 form a directed set.

It is clear that a simplicial decomposition α is completely determined by its 1-simplexes $\rho_1, \cdots, \rho_N$ (for some $N > n$), each p-simplex of α containing precisely p 1-simplexes $\rho_{i_1}, \cdots, \rho_{i_p}$ in its closure. If $\boldsymbol{R}^n$ is endowed with its usual euclidean norm then points on each 1-simplex ρ_i can be identified with their norms $x_i \in \boldsymbol{R}^+$, and points in the open p-simplex determined by $\rho_{i_1}, \cdots, \rho_{i_p}$ can be identified by the coordinates $(x_{i_1}, \cdots, x_{i_p}) \in (\boldsymbol{R}^+)^p$.

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Now define $\varphi: R^+ \to R^+$ by setting $\varphi(x) = \exp \frac{1}{2}(x - x^{-1})$, and observe that φ is a diffeomorphism with inverse φ^{-1} given by $\varphi^{-1}(y) = \ln y + (1 + \ln^2 y)^{1/2}$. One can extend φ to a homeomorphism $\bar{\varphi}: \bar{R}^+ \to \bar{R}^+$ by setting $\bar{\varphi}(0) = 0$. Then there is a diffeomorphism Φ of each open p-simplex of α into itself given by $\Phi(x_{i_1}, \cdots, x_{i_p}) = (\varphi(x_{i_1}), \cdots, \varphi(x_{i_p}))$, and a homeomorphism of the closure into itself given by $\bar{\Phi}(x_{i_1}, \cdots, x_{i_p}) = (\bar{\varphi}(x_{i_1}), \cdots, \bar{\varphi}(x_{i_p}))$. Since $\bar{\varphi}(0) = 0$ it follows that the homeomorphisms $\bar{\Phi}$ agree on intersections of closures of simplexes in α, hence that the diffeomorphisms Φ in all dimensions induce a homeomorphism $R^n \to R^n$. In the following definition we replace the latter homeomorphism by its N-fold composition, where N is the number of 1-simplexes in α, and for convenience we let φ_N denote the N-fold composition of $\varphi: R^+ \to R^+$.

DEFINITION. For the simplicial decomposition α of R^n at 0, with 1-simplexes $\rho_1, \cdots, \rho_N$, the *smoothing homeomorphism* $\Phi_\alpha: R^n \to R^n$ is given by setting $\Phi_\alpha(0) = 0$ and $\Phi_\alpha(x_{i_1}, \cdots, x_{i_p}) = (\varphi_N(x_{i_1}), \cdots, \varphi_N(x_{i_p}))$ for points $(x_{i_1}, \cdots, x_{i_p})$ of open p-simplexes in α with the 1-simplexes $\rho_{i_1}, \cdots, \rho_{i_p}$ on their boundaries.

2. **Smoothable function algebras.** In this section we assign an algebra A of continuous functions $f: M \to R^1$ to each PL manifold M. We begin by taking $M = R^n$, for which we first define the algebra A_0 of germs of elements of A at $0 \in R^n$.

LEMMA. *If $f: R^n \to R^1$ is a piecewise linear function which is linear on each simplex of a simplicial decomposition α of R^n at 0, then $f \circ \Phi_\alpha: R^n \to R^1$ is everywhere smooth.*

The proof of the Lemma depends primarily on the following observation: if $f: R^1 \to R^1$ is any function which is smooth except at $0 \in R^1$, and if the derivatives of f are bounded in a deleted neighborhood of 0, then $\lim_{x \to 0} (f \circ \varphi)^{(q)}(x) = 0$ for each $q > 0$. The details will appear in [6].

The conclusion of the Lemma holds for a very broad class of functions on R^n, including many functions $f: R^n \to R^1$ for which the derivatives of $f \circ \Phi_\alpha$ satisfy no conditions other than smoothness. We let $\Phi_\alpha^{-1}(C^\infty(R^n))$ represent the algebra of those $f: R^n \to R^1$ such that $f \circ \Phi_\alpha \in C^\infty(R^n)$. It will be shown in [6] that if $\beta > \alpha$ then the homeomorphism $\Phi_\alpha^{-1}\Phi_\beta: R^n \to R^n$ is smooth, although its jacobian vanishes on some subset of R^n; a fortiori $\Phi_\alpha^{-1}(C^\infty(R^n)) \subset \Phi_\beta^{-1}(C^\infty(R^n))$.

DEFINITION. A continuous function $f: R^n \to R^1$ is *smoothable at* $0 \in R^n$ if its germ at 0 is the germ of an element of $\Phi_\alpha^{-1}(C^\infty(R^n))$ for some simplicial decomposition α of R^n. A continuous function $f: R^n \to R^1$ is *smoothable at* $P \in R^n$ if $f \circ \tau$ is smoothable at 0 for the translation $\tau: R^n \to R^n$ carrying 0 into P. For any open subset $V \subset R^n$, a continuous function $f: V \to R^1$ is *smoothable on* V if and only if it is smoothable at each $P \in V$.

Trivially, if $f: V \to R^1$ is smooth then it is also smoothable, and the Lemma implies that if $f: V \to R^1$ is piecewise linear then it is also smoothable.

The smoothable functions on any open $V \subset R^n$ form an algebra. For if f and g are smoothable at 0 then the germs of $f \circ \Phi_\alpha$ and $g \circ \Phi_\beta$ are smooth at 0 for some α and β, and so are the germs of $f \circ \Phi_\gamma$ and $g \circ \Phi_\gamma$ for any γ with $\gamma > \alpha$ and $\gamma > \beta$, so that $(f-g) \circ \Phi_\gamma$ and $f \cdot g \circ \Phi_\gamma$ are smooth. In fact, the algebra A_0 of germs of smoothable functions at $0 \in R^n$ is precisely the direct limit $\lim_\alpha \Phi_\alpha^{-1}(C^\infty(R^n))_0$, where the subscript 0 indicates germs at 0, and where each $\Phi_\alpha^{-1}(C^\infty(R^n))_0 \to \Phi_\beta^{-1}(C^\infty(R^n))_0$ is an inclusion homomorphism for $\alpha < \beta$.

We recall that one can define PL manifolds in terms of atlases and PL homeomorphisms of open sets in R^n just as one defines smooth manifolds in terms of atlases and diffeomorphisms of open sets in R^n. (See [8], for example.) Specifically, one covers a PL manifold M with open sets U_i for which there are homeomorphisms $\Psi_i: U_i \to V_i$ onto open sets $V_i \subset R^n$, and the compositions $(\Psi_j | U_i \cap U_j) \circ (\Psi_i | U_i \cap U_j)^{-1}$ are required to be PL homeomorphisms. It will be shown in [6] that the composition of a PL homeomorphism with a smoothable function is smoothable, so that the following definition makes sense:

DEFINITION. For any PL manifold M let $\{U_i\}$ be an atlas with homeomorphisms $\Psi_i: U_i \to V_i$ onto open sets $V_i \subset R^n$ as above. Then the *smoothable function algebra* of M consists of those continuous $f: M \to R^1$ such that $(f| U_i) \circ \Psi_i^{-1}$ is smoothable on V_i for each i.

3. Differential forms in PL. The usual derivation $d: C^\infty(R^n) \to \mathcal{E}(C^\infty(R^n))$ is given by

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n,$$

and it can be described in terms of corresponding derivations of the algebras $C^\infty(R^n)_P$ of germs of $C^\infty(R^n)$ at each $P \in R^n$. This is a general phenomenon about derivations of arbitrary function algebras A: any derivation $d: A \to E$ may be regarded as a section of a sheaf of derivations of the sheaf $\boldsymbol{A}$ of germs A_P of A at points P of the maximal spectrum of A into a sheaf of modules over $\boldsymbol{A}$. (See [5], for example.) We consider only sections over the subset M of the maximal spectrum when A is the smoothable function algebra of M.

Any homomorphism $C^\infty(R^n)_0 \to C^\infty(R^n)_0$ of the germs of smooth functions gives rise to a $C^\infty(\boldsymbol{R}^n)_0$-module homomorphism $\mathcal{E}(C^\infty(\boldsymbol{R}^n)_0)\to\mathcal{E}(C^\infty(\boldsymbol{R}^n)_0)$ which commutes with d. In particular if α and β are simplicial decompositions of $\boldsymbol{R}^n$ at 0 with $\alpha<\beta$ then $\Phi_\alpha^{-1}\Phi_\beta:\boldsymbol{R}^n\to\boldsymbol{R}^n$ is a smooth map which induces a homomorphism $C^\infty(\boldsymbol{R}^n)_0\to C^\infty(\boldsymbol{R}^n)_0$. For convenience we rewrite the resulting commutative diagram

$$\begin{array}{ccc} C^\infty(\boldsymbol{R}^n)_0 & \to & C^\infty(\boldsymbol{R}^n)_0 \\ d\downarrow & & \downarrow d \\ \mathcal{E}(C^\infty(\boldsymbol{R}^n)_0) & \to & \mathcal{E}(C^\infty(\boldsymbol{R}^n)_0) \end{array} \quad \text{in the form} \quad \begin{array}{ccc} \Phi_\alpha^{-1}(C^\infty(\boldsymbol{R}^n)_0) & \to & \Phi_\beta^{-1}(C^\infty(\boldsymbol{R}^n)_0) \\ d_\alpha\downarrow & & \downarrow d_\beta \\ \mathcal{E}_\alpha(A_0) & \to & \mathcal{E}_\beta(A_0) \end{array}$$

to obtain a derivation $d_0:A_0\to\mathcal{E}(A_0)$ of the form

$$\lim_\alpha d_\alpha:\ \lim_\alpha \Phi_\alpha^{-1}(C^\infty(\boldsymbol{R}^n)_0)\to\lim_\alpha \mathcal{E}_\alpha(A_0).$$

Replacing $0\in\boldsymbol{R}^n$ by arbitrary $P\in\boldsymbol{R}^n$ we then induce a derivation $d:A\to\mathcal{E}(A)$ of the smoothable function algebra A on $\boldsymbol{R}^n$. The method of §2 then provides a derivation $d:A\to\mathcal{E}(A)$ of the smoothable function algebra A on any PL manifold M.

4. The de Rham theorem in PL. Since exterior products commute with direct limit, and since the (acyclic) cochain complex $(\Lambda\mathcal{E}(C^\infty(\boldsymbol{R}^n)_0),d)$ is classically defined for the algebra $C^\infty(\boldsymbol{R}^n)_0$ it follows that one can form the exterior algebra $\Lambda\mathcal{E}(A)$ and the cochain complex $(\Lambda\mathcal{E}(A),d)$ for any algebra A of smoothable functions.

THEOREM. *If A is the smoothable function algebra on a PL manifold M, then the homology of $(\Lambda\mathcal{E}(A),d)$ is isomorphic to the real cohomology of M.*

PROOF. It will suffice to establish the PL analog of the Poincaré lemma: the usual sheaf-theoretic argument then applies to the PL case as well as the smooth case. (See [3], for example, which quotes the Poincaré lemma but otherwise invokes no properties of smooth function algebras not also shared by smoothable function algebras; smoothable partitions of unity present no problem.) However, the classical Poincaré lemma states that $(\Lambda\mathcal{E}(C^\infty(\boldsymbol{R}^n)_0),d)$ is acyclic, and we shall use this result to obtain the analogous result that $(\Lambda\mathcal{E}(A_0),d)$ is acyclic for the algebra $A_0=\lim_\alpha\Phi_\alpha^{-1}(C^\infty(\boldsymbol{R}^n)_0)$ of germs of smoothable functions at $0\in\boldsymbol{R}^n$. If

$$\theta=\sum_q f_0^q\,df_1^q\wedge\cdots\wedge df_p^q\in\Lambda^p\mathcal{E}(A_0)\quad\text{for } f_j^i\in A_0$$

then since A_0 is a direct limit there is a simplicial decomposition α of

R^n at 0 for which each $f_j^i \circ \Phi_\alpha$ is smooth. If in addition $d\theta=0$ then $d_\alpha\theta=0$; that is,

$$d(\theta\Phi_\alpha) = d\sum_q g_0^q\, dg_1^q \wedge \cdots \wedge dg_p^q = 0 \in \Lambda^{p+1}\mathcal{E}(C^\infty(R^n)_0),$$

where $g_j^i=f_j^i \circ \Phi_\alpha$. The classical Poincaré lemma then provides

$$\psi = \sum_r h_0^r\, dh_1^r \wedge \cdots \wedge dh_{p-1}^r \in \Lambda^{p-1}\mathcal{E}(C^\infty(R^n)_0) \quad \text{with } d\psi = \theta\Phi_\alpha,$$

where $h_j^i \in C^\infty(R^n)_0$. It follows for $k_j^i=h_j^i \circ \Phi_\alpha^{-1} \in A_0$ and $\psi\Phi_\alpha^{-1} = \sum_r k_0^r\, dk_1^r \wedge \cdots \wedge dk_{p-1}^r \in \Lambda^{p-1}\mathcal{E}(A)$ that $d(\psi\Phi_\alpha^{-1})=\theta$. This completes the Poincaré lemma in PL, hence the de Rham theorem in PL.

5. **Pontrjagin classes in PL.** Let A be the smoothable function algebra of a PL manifold M, and let $\mathfrak{F}$ be a graded left $\Lambda\mathcal{E}(A)$-module. Then $\mathfrak{F}$ is also a right $\Lambda\mathcal{E}(A)$-module with respect to the product $\Phi\theta=(-1)^{p(q+1)}\theta\Phi$ for $\Phi\in\mathfrak{F}^{(p)}$ and $\theta\in\Lambda^q\mathcal{E}(A)$, and one can form the tensor algebra $\otimes_{\Lambda\mathcal{E}(A)}\mathfrak{F}$. Let $\mathfrak{I}\subset\otimes_{\Lambda\mathcal{E}(A)}\mathfrak{F}$ be the two-sided ideal generated by elements $\Phi\otimes\Psi+(-1)^{(p+1)(q+1)}\Psi\otimes\Phi$ for $\Phi\in\mathfrak{F}^{(p)}$ and $\Psi\in\mathfrak{F}^{(q)}$, and let $\Lambda_{\Lambda\mathcal{E}(A)}\mathfrak{F}$ be the quotient of $\otimes_{\Lambda\mathcal{E}(A)}\mathfrak{F}/\mathfrak{I}$. For example, if $\mathfrak{F}=\Lambda\mathcal{E}(A)\otimes\mathcal{E}(A)$ then $\Lambda_{\Lambda\mathcal{E}(A)}\mathfrak{F}=\Lambda\mathcal{E}(A)\otimes\Lambda\mathcal{E}(A)$. If $\mathfrak{F}$ is locally a direct limit of free $\Lambda\mathcal{E}(C^\infty(R^n)_P)$-modules as in the preceding example, then for any two-sided $\Lambda\mathcal{E}(A)$-module homomorphism $K:\mathfrak{F}\to\mathfrak{F}$ one can define $\det K\in\Lambda\mathcal{E}(A)$; in this case $\mathfrak{F}$ *admits determinants.*

A *connection* in a left $\Lambda\mathcal{E}(A)$-module $\mathfrak{F}$ is any real linear map $D:\mathfrak{F}\to\mathfrak{F}$ of degree $+1$ such that $D\theta\Phi=d\theta\cdot\Phi+(-1)^p\theta\cdot D\Phi$ for $\theta\in\Lambda^p\mathcal{E}(A)$. The *curvature* K of D is the composition $D\circ D$, trivially $\Lambda\mathcal{E}(A)$-linear on each side. Here is a very general real Chern-Weil theorem, whose proof will appear in [6]:

PROPOSITION. *If $\mathfrak{F}$ admits determinants then* $\det(I+K/2\pi)$ *is closed for any connection D, and the de Rham cohomology class* $[\det(I+K/2\pi)]$ *is an element of $H^{4*}(M)$ which is independent of D.*

We have already observed that $\Lambda\mathcal{E}(A)\otimes\mathcal{E}(A)$ admits determinants.

LEMMA. $\Lambda\mathcal{E}(A)\otimes\mathcal{E}(A)$ *has at least one connection.*

Now recall from [4] or [7] that the total rational (or real) Pontrjagin class $p(M)$ of a PL manifold M is constructed via the Hirzebruch L-polynomials in such a way that if M happens to carry a smooth structure then $p(M)$ is the Pontrjagin class of the tangent bundle $\tau(M)$. The following construction avoids the L-polynomials; its proof will appear in [6].

THEOREM. *Let A be the smoothable function algebra of a PL manifold M and let $\mathfrak{F}$ be the left $\Lambda\mathcal{E}(A)$-module $\Lambda\mathcal{E}(A)\otimes\mathcal{E}(A)$; then* $[\det(I+K/2\pi)]$ $\in H^{4*}(M)$ *is the Pontrjagin class of M.*

We remark that one can probably establish a PL version of the Gauss-Bonnet theorem within the framework of the present paper, which would more closely parallel the classical formula of [2] than the polyhedral results of [1].

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