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DIRICHLET FINITE SOLUTIONS OF $\Delta u = Pu$, AND CLASSIFICATION OF RIEMANN SURFACES[1]

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1. Problem and its background. Consider a C^1 differential $P(z)\,dx\,dy$ $(z=x+iy)$ on an open Riemann surface R with $P(z) \geqq 0$. We denote by $PX(R)$ the set of C^2 solutions on R of the elliptic equation $\Delta u = Pu$, or more precisely, of $d^*du(z) = u(z)P(z)\,dx\,dy$, with a certain property X. For $P \equiv 0$ we use the traditional notation HX instead of OX. Let O_{PX} be the set of pairs (R, P) such that $PX(R)$ reduces to constants. Instead of $(R, P) \in O_{PX}$ we simply write $R \in O_{PX}$ if P is well understood. As for X we let B stand for boundedness, D for the finiteness of the Dirichlet integral $D_R(u) = \int_R du \wedge {}^*du$, and E for the finiteness of the energy integral $E_R(u) = D_R(u) + \int_R Pu^2\,dx\,dy$; we also consider combinations of these properties. It is known that

$$O_G \subsetneqq O_{PB} \subsetneqq O_{PD} \subset O_{PBD} \subset O_{PE} = O_{PBE}. \tag{1}$$

Here O_G is the class of pairs (R, P) such that there exists no harmonic Green's function on R.

This type of classification problem was initiated by Ozawa [4] in 1952. It first came as a surprise when Myrberg [2] proved in 1954 the unrestricted existence of the Green's function for the equation $\Delta u = Pu$ $(P \not\equiv 0)$ for every R. This also eliminated the need of considering the nonexistence of nonnegative solutions in the case $P \not\equiv 0$. Following Myrberg's discovery, work in this direction largely pursued aspects which were different in nature from those in the harmonic case. Typically classes PD and O_{PD} were first considered by Royden [6] in 1959. Since the energy integral $E(u)$ for $\Delta u = Pu$ plays the same role as the Dirichlet integral $D(u)$ for the harmonic case, it is natural that PE and O_{PE} share properties of HD and O_{HD}. In this

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sense the study of PD and O_{PD} requires an entirely new technique. The author [3] showed in 1961 that every $u \in PD$ can be decomposed into $u = u_1 - u_2$ where $u_i \geqq 0$ and $u_i \in PD$ $(i = 1, 2)$. Therefore the study of PD can be viewed as that of Dirichlet finite subharmonic functions; this is important from the viewpoint of classical potential theory proper.

During the intervening decade rather numerous investigations have been published on this subject, but to the author's knowledge, no explicit further contributions to the theory of classes PD and O_{PD} has been made. One of the central problems, as the author sees it, is to determine whether the inclusion $O_{PD} \subset O_{PBD}$ is strict or not. In the harmonic case and of course in the case of PE, we have the *Virtanen identity* $O_{HD} = O_{HBD}$.

The object of this note is to announce that *we do have the same conclusion* $O_{PD} = O_{PBD}$ despite the fact that PD is quite different in nature from HD.

In passing, we remark that the same is also true if R is replaced by a C^∞ Riemannian manifold with a C^∞ metric tensor g_{ij} of dimension $m \geqq 2$. The proof for the case of Riemann surfaces obviously reproduces verbatim. The result is true even for C^1 manifolds with locally bounded measurable metric tensors g_{ij} and functions P. The proof is again essentially the same as for Riemann surfaces but technically the reasoning is more delicate.

2. **Main result.** Virtanen's proof [8] for $O_{HD} = O_{HBD}$ consists in showing the boundedness of the reproducing kernel for HD viewed as a Hilbert space. It was, essentially, Royden [5] who pointed out that the class HD is a vector lattice and that therefore HBD is dense in HD; this in turn gives $O_{HD} = O_{HBD}$. Our result is rather of the latter nature.

THEOREM. *For any u in $PD(R)$ there exists a sequence $\{u_n\}$ $(n = 1, 2, \cdots)$ in $PBD(R)$ such that* $\sup_R |u_n| = \min(n, \sup_R |u|)$, $u = \lim_n u_n$ *uniformly on each compact set of R, and* $\lim_n D_R(u - u_n) = 0$. *If moreover u is nonnegative, then $\{u_n\}$ can be chosen nondecreasing.*

From this the *Virtanen-type identity*

$$O_{PD} = O_{PBD} \tag{2}$$

immediately follows. We can also show that $PD(R)$ *is a vector lattice.* However neither the Theorem nor (2) is a consequence of this fact since the constant 1 need not be in PD.

The situation can be fully understood only by using the Royden compactification R^* of R (see e.g. [7]). We denote by $\Delta = \Delta(R)$ the harmonic boundary of R, that is, the set of regular points of $\Gamma = R^* - R$ with respect to the harmonic Dirichlet problem. A point z^* in Δ will be called a *P-energy nondensity point* if there exists an open neighborhood U^* of z^* in R^* such that

$$\int_{U\times U} G_U(z, w)P(z)P(w)\, dv(z)\, dv(w) < \infty. \tag{3}$$

Here $U = U^* \cap R$, G_U is the harmonic Green's function on U, and $dv(z) = dx\, dy$ $(z = x + iy)$. The set Δ_P of P-energy nondensity points is open in Δ. Since the functions in PD are continuously extendable to R^* in the extended sense, we may consider PD-functions continuous on R^*. We can show that $PD|\Delta - \Delta_P = \{0\}$. Instead of describing the entire picture of $PD(\Delta_P) = \{u|\Delta_P;\ u \in PD\}$ we only mention the following relation, which gives the essence of our theorem:

$$PD(\Delta_P) \supset \{u\,|\Delta_P;\, u \in HBD(R), \mathrm{Supp}\,(u) \subset \Delta_P\}. \tag{4}$$

In addition to this geometric tool we need an analytic one, the integral operator T_Ω defined by

$$T_\Omega \varphi = -(2\pi)^{-1} \int_\Omega G_\Omega(\cdot, z)\varphi(z)P(z)\, dv(z).$$

Here Ω is an open subset of R with a smooth relative boundary $\partial\Omega$ which may be empty, i.e. $\Omega = R$. For every u in $PD(\Omega)$ we have

$$u = \pi_\Omega u + T_\Omega u, \qquad D_\Omega(u) = D_\Omega(\pi_\Omega u) + D_\Omega(T_\Omega u) \tag{5}$$

where $\pi_\Omega u$ is the harmonic projection of u (cf. [7]). Moreover

$$D_\Omega(T_\Omega u) = (2\pi)^{-1} \int_{\Omega\times\Omega} G_\Omega(z, w)u(z)u(w)P(z)P(w)\, dv(z)\, dv(w), \tag{6}$$

with all integrals understood in the sense of Lebesgue. These relations are easy consequences of the Stokes formula, a standard exhausting method, and the fact that a function u in PD is a difference of two nonnegative PD-functions (cf. [3]). We also have

$$T_\Omega u \,|\, (\partial\Omega) \cup (\bar{\Omega} \cap \Delta) = 0. \tag{7}$$

3. **Sketch of the proof.** We present an outline of the proof only for $O_{PD} = O_{PBD}$, since this identity gives the essense of our results. We may assume $P \not\equiv 0$. Suppose there is a nonconstant u in $PD(R)$. By (6) it can be seen that there exists a point z^* in Δ belonging to Δ_P.

Let U be the corresponding open set in (3). We may modify U to have a smooth ∂U. Choose an h in $HBD(U)$ such that $h|\partial U=0$, $0\leqq h\leqq 1$ on U, and $h(z^*)=1$. Again by the standard exhausting method, we see that the integral equation of the Fredholm type

$$(I - T_U)u = h$$

has a unique solution u on U which is in $PD(U)$, with I the identity operator. Here the condition (3) is essential, and (6) is also employed. We deduce from (7) that $u|\partial\Omega=0$, $u(z^*)=1$, and $0\leqq u\leqq h\leqq 1$ on $\overline{U}$. If we extend u to R by setting $u=0$ on $R-U$, then u is a Dirichlet finite subsolution of $\Delta u=Pu$. Therefore we can construct a v in $PBD(R)$ such that $u\leqq v\leqq 1$ on R and a fortiori $R\notin O_{PBD}$. Here we have again used the exhausting method and the following entirely obvious, once observed, but useful fact (cf. [3]):

Weak Dirichlet principle. Let Ω be a regular subregion of R and $\mathfrak{F}_\varphi$ the class of Dirichlet finite subsolutions $v\geqq 0$ of $\Delta u=Pu$ on Ω with continuous boundary values φ at $\partial\Omega$. Then the variational problem $\min_{v\in\mathfrak{F}_\varphi} D_\Omega(v)$ has a unique solution u which is in $\mathfrak{F}_\varphi\cap PBD(\Omega)$.

4. **Additional remarks.** From the proof one sees at once that *in the definition of a P-energy nondensity point the function G_U may be replaced by G_R.* Moreover, *$R\notin O_{PD}$ if and only if there exists a subregion U of R with a smooth ∂U such that $U\notin SO_{HD}$ and U satisfies* (2) (cf. [7] for SO_{HD}). This may be viewed as a counterpart of the Bader-Parreau-Mori two domain criterion for an R not in O_{HD} (one domain criterion!). Of course the above statement is a restatement of the fact that *$R\in O_{PD}$ if and only if $\Delta_P=\emptyset$.*

The revised string of inclusion relations (1) now reads:

$$O_G \underset{\neq}{\subset} O_{PB} \underset{\neq}{\subset} O_{PD} = O_{PBD} \subset O_{PE} = O_{PBE}. \tag{8}$$

The only important open problem in this context is to prove or disprove the strictness of the inclusion $O_{PD}\subset O_{PE}$. At this point we must quote the recent important contributions mainly to the class PE by Glasner and Katz [1], who introduced the notion (not the term) of a *P-nondensity point* for points z^* in Δ characterized by

$$\int_U P(z)\, dv(z) < \infty \tag{9}$$

instead of (3). The set Δ^P of such points relates to O_{PE} in the following fashion: $R\in O_{PE}$ if and only if $\Delta^P=\emptyset$. Clearly

$$\Delta^P \subset \Delta_P. \tag{10}$$

Therefore the problem may be rephrased as follows: Does $\Delta^P = \emptyset$ imply $\Delta_P = \emptyset$ or not?

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ADDED IN PROOF. We found $O_{PD} \subsetneq O_{PE}$ (M. Nakai, *A remark on classification of Riemann surfaces with respect to* $\Delta u = Pu$, Bull. Amer. Math. Soc. **77** (1971), (to appear)).