HILBERT CUBE MANIFOLDS

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1. **Introduction.** It is the purpose of this note to announce some new results concerning *Hilbert cube manifolds* (or *Q-manifolds*), i.e. separable metric spaces which have open covers by sets homeomorphic to open subsets of the Hilbert cube, I^∞. Their proofs will appear in a longer paper that is in preparation [5].

These results parallel a number of embedding, characterization, and homeomorphism theorems that have been established recently for paracompact manifolds modeled on various infinite-dimensional linear spaces (see [4] for a partial summary and [6], [8] for more recent generalizations).

In obtaining these Q-manifold results the linear space apparatus used in some of the corresponding results of [4], [6], and [8] could not be used. Thus in most cases new techniques had to be devised. We list these results below along with some of the principal results on Q-manifolds that have been established elsewhere. We also list a number of open questions.

2. **Definitions and notation.** We represent I^∞ as the countable infinite product of closed intervals $[-1, 1]$ and we let $0=(0, 0, \cdots)\in I^\infty$.

Following Anderson [1] we say that a closed subset K of a topological space X has *Property* Z in X provided that for each nonnull and homotopically trivial (i.e. all homotopy groups are trivial) open subset U of X, $U\backslash K$ is nonnull and homotopically trivial. We also call K a *Z-set*.

Let X and Y be topological spaces and let $\mathcal{U}$ be an open cover of Y. Then functions $f, g: X\rightarrow Y$ are said to be $\mathcal{U}$-*close* provided that for each $x\in X, f(x)$ and $g(x)$ lie in some element of $\mathcal{U}$. A function $h: Y\rightarrow Y$ is said to be *limited by* $\mathcal{U}$ provided that for each $y\in Y$, y and $h(y)$ lie in some element of $\mathcal{U}$. A function $H: X\times[0, 1]\rightarrow Y$ is also said to be limited by $\mathcal{U}$ provided that for each $x\in X$, $H(\{x\}\times[0, 1])$ lies in some element of $\mathcal{U}$. By $\mathrm{St}^n(\mathcal{U})$ we mean the nth star of the cover $\mathcal{U}$, defined in the usual manner.

An isotopy $F: X\times[0, 1]\rightarrow Y$ is said to be an *invertible ambient*

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isotopy provided that F is ambient (i.e. each level is onto) and $F^*: X \times [0, 1] \to Y \times [0, 1]$, defined for each $t \in [0, 1]$ by $F^*(x, t) = (F(x, t), t)$, is a homeomorphism.

A subset K of a topological space X is said to be *bicollared* provided that there is an open embedding $h: K \times (-1, 1) \to X$ such that $h(x, 0) = x$, for all $x \in K$.

A (topological) *polyhedron* is a space homeomorphic ($\cong$) to $|K|$, for some countable locally-finite simplicial complex K.

For topological spaces X and Y, a continuous function $f: X \to Y$ is said to be *proper* provided that given any compact subset K of Y, $f^{-1}(K)$ is compact. If X and Y have the same homotopy type, then we say that X and Y have the same *proper homotopy type* provided that the homotopies involved in the usual definition are proper maps. We remark that proper maps and proper homotopies have proved to be useful in investigating Q-manifolds.

3. Topological stability.

THEOREM 1 [3]. *If X is a Q-manifold, then $X \times I^\infty \cong X$. In particular, $X \times [0, 1] \cong X$.*

4. Product theorems.

THEOREM 2 [7]. *If P is a compact contractible polyhedron, then $P \times I^\infty \cong I^\infty$.*

COROLLARY 1 [7]. *If P is any polyhedron, then $P \times I^\infty$ is a Q-manifold.*

QUESTION 1. *If X is a compact metric AR, then is $X \times I^\infty \cong I^\infty$?*

5. Factoring theorems.

THEOREM 3 [5]. *If X is an open subset of I^∞, then there is a polyhedron P such that $X \cong P \times I^\infty$.*

In Theorem 4 and many of the subsequent theorems we will have to consider Q-manifolds with half-open interval factors $[0, 1)$. The factor $[0, 1)$ has the effect of smoothing out anomalies that appear in Q-manifolds, and in most cases the results are false if this factor is omitted. It is interesting to note that an open interval factor $(0, 1)$ will not suffice in all cases, for example in Theorem 6.

THEOREM 4 [5]. *If X is a Q-manifold, then there is a polyhedron P such that $X \times [0, 1) \cong P \times I^\infty$.*

COROLLARY 2 [5]. *If X is a Q-manifold, then there is a polyhedron P and a Z-set $F \subset X$ such that $X \setminus F \cong P \times I^\infty$.*

QUESTION 2. *If X is a Q-manifold, then is there a polyhedron P such that $X \cong P \times I^\infty$?*

6. Characterization theorems.

THEOREM 5 [7]. *Let K and L be countable locally-finite simplicial complexes such that L is a formal deformation of K (in the sense of Whitehead [9]). Then $|K| \times I^\infty \cong |L| \times I^\infty$.*

THEOREM 6 [5]. *Let X and Y be Q-manifolds and let $f: X \to Y$ be a homotopy equivalence. Then $f \times \mathrm{id}: X \times [0, 1) \to Y \times [0, 1)$ is homotopic to a homeomorphism of $X \times [0, 1)$ onto $Y \times [0, 1)$.*

(Note that I^∞ and $I^\infty \setminus \{0\}$ are Q-manifolds of the same homotopy type which are not homeomorphic.)

COROLLARY 3 [5]. *If X and Y are Q-manifolds which have the same homotopy type, then there are Z-sets $F \subset X$ and $K \subset Y$ such that $X \setminus F \cong Y \setminus K$.*

THEOREM 7 [5]. *If P and R are polyhedra which have the same homotopy type, then $P \times (I^\infty \setminus \{0\}) \cong R \times (I^\infty \setminus \{0\})$.*

QUESTION 3. *If X and Y are Q-manifolds which have the same proper homotopy type, then is $X \cong Y$?*

7. Open embedding of Q-manifolds.

THEOREM 8 [5]. *If X is a Q-manifold, then $X \times [0, 1)$ can be embedded as an open subset of I^∞.*

(Note that $S^1 \times I^\infty$ is a Q-manifold which cannot be embedded as an open subset of I^∞, where S^1 is the 1-sphere.)

COROLLARY 4 [5]. *If X is a Q-manifold, then we can write $X = U \cup V$, where U and V are open subsets of X which can be embedded as open subsets of I^∞.*

QUESTION 4. *Let X be a Q-manifold which has the homotopy type of a compact polyhedron P. Then is there a Z-set $F \subset P \times I^\infty$ such that $X \cong (P \times I^\infty) \setminus F$?*

Of particular interest is a special case of Question 4.

QUESTION 4′. *Let X be a contractible Q-manifold. Then is there a Z-set $F \subset I^\infty$ such that $X \cong I^\infty \setminus F$?*

8. Compact Q-manifolds.

THEOREM 9 [5]. *Let X be a compact Q-manifold which has the* *homotopy type of a compact polyhedron P. Then there is a copy P' of P which is a Z-set in X such that $X\backslash P' \cong P \times (I^\infty \backslash \{0\})$.*

THEOREM 10 [5]. *Let X be a compact Q-manifold. A necessary and sufficient condition that X be homeomorphic to I^∞ is that X be homotopically trivial.*

THEOREM 11 [5]. *Let X be a compact Q-manifold which has the homotopy type of a compact polyhedron P. Then there is an embedding $h: X \to I^\infty$ such that $\mathrm{Bd}(h(X)) \cong P \times I^\infty$ and $\mathrm{Cl}(I^\infty \backslash h(X)) \cong I^\infty$.*

QUESTION 5. *If X and Y are compact Q-manifolds which have the same homotopy type, then is $X \cong Y$?*

(Note that in the compact case, the concepts of homotopy type and proper homotopy type coincide.)

9. Property Z.

THEOREM 12 [2]. *If X is a Q-manifold and $F \subset X$ is a Z-set, then there is a homeomorphism $h: X \to X \times I^\infty$ such that $h(x) = (x, 0)$, for all $x \in F$.*

10. Mapping replacement theorems.

THEOREM 13 [2]. *Let X be a Q-manifold, $\mathcal{U}$ be an open cover of X, A be a locally compact separable metric space, and let $f: A \to X$ be a proper map. Then there exists an embedding $g: A \to X$ such that $g(A)$ is a Z-set in X and g is $\mathrm{St}(\mathcal{U})$-close to f. Moreover we can choose g homotopic to f.*

If we assume the existence of a $[0, 1)$-factor of X, then we can drop the requirement that f be proper, but we sacrifice the "closeness" of g to f.

THEOREM 14 [2]. *Let X be a Q-manifold, A be a locally compact separable metric space, and let $f: A \to X \times [0, 1)$ be a continuous function. Then there exists an embedding $g: A \to X \times [0, 1)$ such that $g(A)$ is a Z-set in $X \times [0, 1)$. Moreover we can choose g homotopic to f.*

11. Replacing homotopies by isotopies.

THEOREM 15 [2]. *Let X be a Q-manifold, A be a locally compact separable metric space, and let f and g be embeddings of A into X such that $f(A)$ and $g(A)$ are Z-sets. Then there exists an invertible ambient isotopy G of X onto itself such that $G_0 = \mathrm{id}$ and $G_1 \circ f = g$ if and only if f and g are properly homotopic. Moreover, if F is a proper homotopy between f and g and $\mathcal{U}$ is an open cover of X for which F is limited by $\mathcal{U}$, then G may be chosen so that it is limited by $\mathrm{St}^4(\mathcal{U})$.*

Once more if we assume the existence of a $[0, 1)$-factor, then we can drop the requirement that the homotopy be proper.

THEOREM 16 [2]. *Let X be a Q-manifold, A be a locally compact separable metric space, and let f and g be embeddings of A into $X\times[0, 1)$ such that $f(A)$ and $g(A)$ are Z-sets. Then there is an invertible ambient isotopy G of X onto itself such that $G_0=\mathrm{id}$ and $G_1\circ f=g$ if and only if f and g are homotopic.*

12. Schoenfliess-type results.

THEOREM 17 [10]. *Let $f, g: I^\infty \to I^\infty$ be embeddings such that $f(I^\infty)$ and $g(I^\infty)$ are bicollared. Then there is a homeomorphism $h: I^\infty\to I^\infty$ such that $h\circ f=g$.*

THEOREM 18 [5]. *Let X, Y be Q-manifolds and let $f, g: X\to Y$ be closed embeddings which are homotopy equivalences and for which $f(X)$, $g(X)$ are bicollared. Then there is a homeomorphism $h: Y\times[0, 1) \to Y\times[0, 1)$ such that $h\circ(f\times\mathrm{id})=g\times\mathrm{id}$, where* id *is the identity mapping on* $[0, 1)$.

QUESTION 6. *Let X, Y be Q-manifolds and let $f, g: X\to Y$ be embeddings which are proper homotopy equivalences and for which $f(X)$, $g(X)$ are bicollared. Then is there a homeomorphism $h: Y\to Y$ for which $h\circ f=g$?*

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