KERNEL FUNCTIONS AND PARABOLIC LIMITS FOR THE HEAT EQUATION

BY JOHN T. KEMPER

Communicated by Wolfgang Wasow, May 4, 1970

Let $D\subset\{(x,t):t>0\}$ be a domain of the plane bounded by curves $x=\eta_1(t)$, $x=\eta_2(t)$, and $t=0$, where $\eta_1(t)<\eta_2(t)$ for all t and, for each $T\in(0,\infty)$, $\eta_i(t)$ satisfies a Lipschitz condition with exponent $\frac{1}{2}$ on the interval $[0, T]$, $i=1, 2$. Let $(X, T)\in D$ and $(y_0, s_0)\in\partial D$ with $s_0<T$. A kernel function for the heat equation in D at (y_0, s_0) with respect to (X, T) is a nonnegative solution of the heat equation in D, $K(x, t)$, which vanishes continuously on $\partial D-\{(y_0, s_0)\}$ and is normalized by the requirement that $K(X, T)=1$.

The notion of a kernel function has been studied in the case of harmonic functions in Lipschitz domains in $\boldsymbol{R}^n$ by Hunt and Wheeden [3], whose results include a representation theorem for nonnegative harmonic functions and a proof of the almost everywhere (with respect to harmonic measure) existence of finite nontangential boundary values for harmonic functions having a one-sided bound in a Lipschitz domain. (See also [2].) The present note describes analogous results for the heat equation in regions of the plane.

THEOREM. *If $(X, T)\in D$ and $(y, s)\in\partial D$ with $s<T$, then there exists a unique kernel function for the heat equation in D at (y, s) with respect to (X, T).*

It is clear that, for $s<T$, a kernel function at (y, s) with respect to (X, T) is completely determined by its values in $D_T=\{(x, t)\in D:t<T\}$. Thus, it suffices to consider kernel functions at (y, s) in the bounded region D_T. One is led to the following representation result.

THEOREM. *Let $\partial_p D_T$ denote the parabolic boundary of D_T, which is $\partial D_T\cap\{(x, t):t<T\}$, and for $(y, s)\in\partial_p D_T$, let $K(x, t, y, s)$ denote the value at (x, t) of the kernel function at (y, s) with respect to (X, T). If $u(x, t)$ is any nonnegative temperature in D_T, then there exists a unique regular Borel measure μ on $\partial_p D_T$ such that*

$$u(x, t) = \int_{\partial_p D_T} K(x, t, y, s)\,d\mu(y, s).$$

AMS 1970 *subject classifications*. Primary 35K05; Secondary 35C15.

Key words and phrases. Heat equation, kernel function, parabolic limits, representation of positive temperatures.

As was seen by Hunt and Wheeden and, previously, by Carleson [1], in the case of harmonic functions, such a representation leads to a statement of existence of certain limits at the boundary. In the harmonic case these are nontangential limits. For the domain D considered here, a function $u(x, t)$ has parabolic limit L at a point $(y, s) \in \partial D$ if $u(x, t)$ has limit L at (y, s) inside any parabolic cone contained in D with vertex at (y, s). (A parabolic cone is a set of the form either $\{(x, t): \alpha|t-s_0|^{1/2} < x-y_0 < \beta\}$ or $\{(x, t): \alpha|x-y_0|^2 < t-s_0 < \beta\}$, depending on whether the vertex (y_0, s_0) is on the "side" or the "bottom" of D. Both α and β are positive.)

We use the term "caloric measure" to denote the measure on D which corresponds to harmonic measure in the case of Laplace's equation.

THEOREM. *If $u(x, t)$ is a solution of the heat equation in D_T such that $u(x, t)$ is nonnegative (or has a one-sided bound in D_T), then $u(x, t)$ has finite parabolic limits on $\partial_p D_T$ except for a set of zero caloric measure.*

A similar theorem can be proven, assuming only that $u(x, t)$ has a one-sided bound in some parabolic cone at each point of $\partial_p D_T$.

Furthermore, all of these theorems can be extended to domains D of the form $D = \{(x, t): t > 0, x > \eta(t)\}$, where $\eta(t)$ again satisfies a Lipschitz condition with exponent $\frac{1}{2}$ on intervals $[0, T]$ with $T < \infty$. Proofs of all the results mentioned here can be found in [4].

REFERENCES

1. L. Carleson, *On the existence of boundary values for harmonic functions of several variables*, Ark. Mat. 4 (1962), 393–399. MR 28 #2232.

2. R. A. Hunt and R. L. Wheeden, *On the boundary values of harmonic functions*, Trans. Amer. Math. Soc. 132 (1968), 307–322. MR 37 #1634

3. ———, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. 147 (1970), 507–527.

4. J. T. Kemper, *Kernel functions and parabolic limits for the heat equation*, Thesis, Rice University, Houston, Texas, 1970.

RICE UNIVERSITY, HOUSTON, TEXAS 77001