EQUIVARIANT DYNAMICAL SYSTEMS

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In this note we consider equivariant vector fields and diffeomorphisms and present results which generalise some well-known theorems of the theory of dynamical systems as developed by Smale and others. The main result is a generalisation of the Kupka-Smale density theorem. Proofs will be given elsewhere; the note is a summary of the author's Ph.D. thesis, done at the University of Warwick.

For a survey of dynamical systems theory, see [1]; for elementary facts about equivariant vector fields, see [2].

M will always denote a compact C^∞ manifold, without boundary, and G a compact Lie group acting differentiably on M. Let $GVB(M)$ and $GFB(M)$ respectively denote the categories of C^∞ G-vector bundles and C^∞ G-fiber bundles over M; we assume paracompact fiber.

Thus $TM \in GVB(M)$ in a natural way. For $E \in GVB(M)$ or $GFB(M)$ we may consider $C^r_G(E) = \{X \in C^r(E) : gXg^{-1}(x) = X(x), g \in G, x \in M\}$: The space of equivariant sections of E. It is well known that for $E \in GVB(M)$, $C^r_G(E)$ is a Banach splitting subspace of $C^r(E)$, with respect to the C^r topology on $C^r(E)$.

As a straightforward generalisation of the proof given for $G = \mathrm{id}$ in [3], we have:

THEOREM 1. *If $E \in GFB(M)$, then $C^r_G(E) \subset C^r(E)$, as a closed C^∞ Banach submanifold, in a natural unique way.*

COROLLARY 1.1. $\mathrm{Diff}^r_G(M) \subset C^r(M, M)$, $r \geqq 1$, *as a C^∞ submanifold. Here* $\mathrm{Diff}^r_G(M)$ *denotes the set of C^r equivariant diffeomorphisms of M, with the C^r topology.*

If $X \in C^r_G(TM)$ and $X(x) = 0_x$, then $X(gx) = 0_{gx}$, $g \in G$, and $G(x)$ is a singular set for X. Similarly for $f \in \mathrm{Diff}^r_G(M)$, with $fx = x$.

Let q be a closed orbit of $X \in C^r_G(TM)$. We define G_q, the "isotropy group of q", by:

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$$G_q = \{g \in G : g(q) = q\}.$$

Then G_q is a closed subgroup of G and, if $x \in q$, G_x (the isotropy group of x) is contained in G_q as a closed normal subgroup.

We may easily show that there are two types of orbits:

1. $G_q/G_x \cong C^k$, where C^k is the cyclic group of order k.

2. $G_q/G_x \cong S^1$, which implies, in particular, that $G(q) = G(x)$ and that q is a C^∞ closed orbit of X.

We now recall a definition of [4], adapted here to the equivariant situation:

DEFINITION 1. Let V be a C^1 compact submanifold of M and let $f \in \mathrm{Diff}_G^r(M)$. We suppose that V is both G-invariant and f-invariant and that M is a Riemannian manifold.

We say f is "G,r-normally hyperbolic on V" iff the tangent bundle of M, restricted to V, splits into three continuous subbundles:

$$T_V M = TV \oplus N^s \oplus N^u,$$

invariant by the differential of f, Tf, and s.t.:

1. $\sup_{x\in V}\|Tf^k|N_x^s\| < \inf_{x\in V}(m(Tf^k|T_xV))$ $1 \leqq k \leqq r$, and Tf contracts N^s.

2. $\inf_{x\in V}(m(Tf^k|N_x^u)) > \sup_{x\in V}\|Tf^k|T_xV\|$ $1 \leqq k \leqq r$, and Tf expands N^u.

Here $m(A) = \inf\{|Ax| : |x| = 1\}$.

Essentially the normal behaviour of Tf is hyperbolic and it dominates the tangent to V behaviour.

As an easy consequence of the definition, we may assume that M is G-Riemannian and that the splitting induced by Tf is G-invariant.

For C^r equivariant flows $\{f^t\}$ we have a similar definition with f^t, for some $t > 0$, replacing f in the above.

DEFINITION 2.

Case A: If $G(x)$ is a singular set for $X \in C_G^r(TM)$, we say $G(x)$ is a 1-generic singular set for X iff $G(x)$ is G,r-normally hyperbolic for X(i.e. the flow of X). Similarly for diffeomorphisms.

Case B: If q is a closed orbit, we say X is 2-generic on $G(q)$ iff $G(q)$ is G,r-normally hyperbolic for X–we note the difference between Type 1 and Type 2 closed orbits.

We define a generalised Poincaré map, for a closed orbit of Type 1, by taking a C^r G normal bundle of $G(q)$, restricting to $G(x)$, $x \in q$, and projecting into M using the exponential map to define a transverse section to the orbit through x, the Poincaré map so defined is C^c equivariant with a fixed set $G(x)$, denote the map by f_X.

Then f_X is 1-generic on $G(x)$ iff X is 2-generic on $G(q)$.

If $H \subset G$ is a subgroup, we let $N(H)$ denote the normaliser of H in G.

If, in addition in Case A, $\dim(N(G_x)) = \dim G_x$, we say that $G(x)$ is a 1*-generic fixed set for X, or for f, if f is a diffeomorphism.

If in Case B, q is an orbit of Type 1 and f_X is 1*-generic, we say that $G(q)$ is a 2*-generic set for X. If q is an orbit of Type 2, we say it is 2*-generic iff rank $(N(G_x)/G_x) = 1$.

These additional conditions essentially imply that under peturbation closed orbits are stable, i.e. do not perturb into noncompact orbits.

THEOREM 2. *With the above notation, the set of equivariant vector fields, satisfying the property that all singular points are* 1*-*generic and all closed orbits are* 2*-*generic, is a residual subset of* $C_G^r(TM)$.

The proof of this result is long and we do not attempt a summary here. We have a similar theorem for diffeomorphisms.

If q is a 2-generic closed orbit of X, we have global stable and unstable manifolds for $G(q)$; in fact we have:

THEOREM 3. *If* $\overline{N}^s$ *is a* C^r G *vector bundle approximation to* N^s *(see Definition* 1) *then there exists a* C^r *injective equivariant immersion* $I: \overline{N}^s \to M$, *s.t.*: $I(\overline{N}^s) = W^s(G(q))$ *and also* $I(\overline{N}^s | q) = W^s(q)$.

Similarly for the unstable manifold of $G(q)$ and for 1-generic singular sets.

REMARK.

1. Such G-vector bundle approximations can be shown to exist.
2. $W^s(G(q))$ is characterised by:

$$W^s(G(q)) = \{z \in M : F_t z \to G(q) \quad \text{as } t \to \infty\};$$

similarly for the other sets.

If $W \subset M$, and $H \subset G$ as a closed subgroup, we set $W_H = \{z \in W : G_z = H\}$.

Let $G(p)$ and $G(q)$ be two generic critical sets of $X \in C_G^r(TM)$, we say $W^s(G(p))$ meets $W^u(G(q))$ "weak G transversally at $y \in M$" iff:

$$(W^s(G(p)))_{G_y} \pitchfork_y (W^u(G(q)))_{G_y} \subset M_{G_y},$$

i.e. we require that the intersection is transversal at y *in* M_{G_y}.

We say $W^s(G(p))$ meets $W^u(G(q))$ weak G transversally iff all points of intersection are weak G transversal.

We say $X \in C_G^r(TM)$ is 3-generic iff X is 1* and 2* generic and the stable and unstable manifolds of generic critical elements of X are weak G transversal. We have:

THEOREM 4. *$C_G^r(TM)$ contains a residual subset of 3-generic vector fields.*

Whilst weak G transversality is the strongest transversality condition one can reasonably ask for in this context, nevertheless difficulties arise as, even if $W \subset M$ is compact $W_H \subset M$, for $H \subset G$, will, in general, be noncompact.

As an easy consequence of work in [4], we have:

THEOREM 5 (GENERALISED EQUIVARIANT HARTMAN'S THEOREM). *If p is a generic critical element of $X \in C_G^r(TM)$, we have an equivariant conjugacy h between $\{F_t^X\}$ and $\{NF_t^X\}$ in a nbd. of $G(p)$. Here $\{F_t^X\}$ denotes the flow of X, and $NF_t^X = TF_t^X | (N^s \oplus N^u)$. h is a homeomorphism.*

Similarly for diffeomorphisms.

We end with a brief consideration of some rather more specific problems.

If $f \in \mathrm{Diff}_G^r(M)$, we say f is a "G-Anosov map" iff:

$$f_*: C_G^0(TM) \to C_G^0(TM);\ X \mapsto Tf \cdot X \cdot f^{-1},$$

is hyperbolic.

As an easy generalisation of the proof for $G = \mathrm{id}$, using Corollary 1.1, we have:

THEOREM 6. *G-Anosov maps are (equivariantly) structurally stable.*

If $X \in C_G^r(TM)$ is s.t.:

1. $\Omega(X)$, mod G, is finite.
2. $\Omega(X)$ consists of families of closed orbits and fixed points.
3. X is 3-generic.

We say X is a "G-Morse-Smale vector field". We could weaken condition 3 to only require 1 and 2 genericity together with weak G transversality of stable and unstable manifolds; we have, with this weaker definition:

THEOREM 7. *Every compact G manifold admits a G-Morse-Smale vector field.*

With a suitable topology on $C_G^r(TM)$ and a suitable definition of G-structural stability which, for reasons of space, we will not reproduce here, one may reasonably ask:

Problem. Are G-Morse-Smale systems structurally stable?

Suffice it to say that the main problem here appears to arise from the fact that in the G-orbit decomposition of M, into sets of the same type—see [5]—some of these sets are noncompact.

References

1. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–819. MR **37** #3598.

2. A. G. Wasserman, *Equivariant differential topology*, Topology **8** (1969), 127–150.

3. R. S. Palais, *Foundations of global non-linear analysis*, Benjamin, New York, 1968.

4. C. C. Pugh, M. W. Hirsch and M. Shub, *Invariant manifolds* (to appear).

5. R. S. Palais, in A. Borel, "*Seminar on transformation groups*," Ann. of Math. Studies, no. 46, Princeton Univ. Press, Princeton, N. J., 1960. MR **22** #7129.

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