# THE ORDER OF THE IMAGE OF THE $J$-HOMOMORPHISM

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ABSTRACT. This note announces a proof of the order of the image of the $J$-homomorphism and gives several other results in homotopy theory which are consequences of the proof.

The set $\Omega^n S^n$ can be identified with the set of all base point preserving maps of $S^n$ into itself. $SO(n)$, acting on $S^n$ as $R^n$ with a point at infinity, is also a set of base point preserving maps of $S^n$ onto itself. This defines $SO(n) \subset \Omega^n S^n$. The induced map in homotopy is called the $J$-homomorphism. If we allow $n$ to go to infinity we have the stable $J$-homomorphism. By Bott's results [3] $\pi_j(SO) = Z$, $j \equiv -1$ mod 4, and $= Z_2 j \equiv 0$, 1 mod 8, $j > 0$, and zero otherwise. Adams [1] showed that the $Z_2$ summand maps monomorphically and Milnor and Kervaire [6] showed that the $Z$ group in dimension $4j-1$ maps nontrivially and its image generates a subgroup of at least a certain order $\lambda_j'$. Adams [1] showed that the order was either $\lambda_j'$ or $2\lambda_j'$ and if $j \equiv 1$ (2) it was $\lambda_j'$. Thus only the two primary part is in question and there only for $j \equiv 0$ (2). Let $\lambda_j$ be the two primary part of $\lambda_j'$. If $4j \equiv 2^{\rho(j)} \bmod 2^{\rho(j)+1}$ (which defines $\rho(j)$) then $\lambda_j = 2^{\rho(j)+1}$. We prove:

THEOREM 1. *The 2-primary order of the image of $J$ in stem $4j-1$ is $\lambda_j$.*

The proof has several corollary results which have some interest. The first result is rather technical but still has some interest. The naming of elements in $H^{**}(A)$ is that given in [5].

THEOREM 2. *The elements $P^i c_0$, $P^i h_1 c_0$, $i \geqq 1$, $P^i h_2$, $i \geqq 1$, in $H^{**}(A)$ represent the image of $J$ in dimension $j \equiv 0$, 1, 3 mod 8. In dimension $8j-1$ the "tower" which ends at the "Adams edge" represents the image of $J$ in that dimension.*

These elements were known to have the desired $e$-invariant property [1] and were believed to be in $J$. Their Whitehead product behavior has been investigated ([2] and [4], for example).

Let $M = Z_2 + Z_2$ (be the module over $A$ with one generator; $\mu$ in

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dimension zero and $Sq^1 \neq 0$. Let $P(x_1, \cdots)$ be a polynomial algebra on generators $x_i$ with bidegree $(2, 2^{i+2}+1)$. Consider the differential $d(x_i) \to x_{i-1}^2 x_1$ in $P$. Let $H(d)$ be the homology under $d$ and $B(d) = \operatorname{im} d$.

For $\alpha \in P$ let the bidegree of $\alpha$ be $(\alpha_s', \alpha_t')$. We will be only interested in the values of $\alpha_s'$ modulo 4 and $\alpha_t'$ modulo 12 so take $(\alpha_s, \alpha_t)$ so that $\alpha_s \equiv \alpha_s'$ (mod 4), $\alpha_t \equiv \alpha_t'$ (mod 4) but $5\alpha_s < \alpha_t - \alpha_s$.

THEOREM 3. *If* $5s \geqq t-s+\epsilon$ *where* $\epsilon$ *depends on the congruence class of* $s$ mod 4 *and* $\epsilon \leqq 6$, *then*

$$\operatorname{Ext}_A^{s,t}(M, Z_2) = \sum_{\alpha \in H(d)} \operatorname{Ext}_{A_0}^{s-\alpha_s, t-\alpha_t}(M \otimes A//A_1, Z_2)$$

$$\oplus \sum_{\alpha \in B(d)} \operatorname{Ext}_A^{s-\alpha_s, t-\alpha_t}(M \otimes A/A(Sq^3, Sq^1), Z_2).$$

COROLLARY 4. *If* $Q$ *is an* $A$ *module which is free over* $A_1$, *the subalgebra generated by* $Sq^1$ *and* $Sq^2$, *then* $\operatorname{Ext}_A^{s,t}(Q, Z_2) = 0$ *for* $5s \geqq t-s+\epsilon$.

THEOREM 5. *Let* $X$ *be a space in the stable category so that* $\Sigma X = RP^2$. *If* $E_r(X)$ *is the Adams spectral sequence converging to* $\pi_*^S(X)$, *then then* $E_5^{s,t} = E_\infty^{s,t}(X) = 0$ *for* $5s \geqq t-s+\epsilon$ *unless*

$$\begin{aligned} s &= 4k, & t-s &= 8k,\ 8k+1,\ 8k+2, \\ &= 4k+1, & t-s &= 8k+1,\ 8k+2,\ 8k+3, \\ &= 4k+2, & t-s &= 8k+2,\ 8k+3,\ 8k+7, \\ &= 4k+3, & t-s &= 8k+4,\ 8k+8,\ 8k+9, \end{aligned}$$

in which cases the groups are $Z_2$.

These elements represent the generators of the image of $J$ and $\mu_j$ [1] on the bottom cell and the elements of order two in the im $J$ and $\mu_j$ coextended on the top cell.

THEOREM 6. *There is a space* Im $J$ *and a map* $f: S^0 \to \operatorname{Im} J$ *so that* $f_*$ *maps the image of* $J$ *and the* $\mu$*'s monomorphically onto the homotopy of* Im $J$.

In [1] a map $f: \Sigma^8 X \to X$ which represents an extension of a coextension of $8\sigma$ is studied. There it is proved that all iterations of $f$ are essential.

THEOREM 7. *If* $\alpha: S^k \to X$ *is a stable map then*

$$S^{k+8j} \xrightarrow[\Sigma^{8j}\alpha]{} \Sigma^{8j} X \xrightarrow[f^j]{} X$$

*is inessential for some j unless $\alpha$ is in one of the classes given by Theorem* 5.

**Some comments on the proof.** Let the spectrum *bo* be the connected *BO* spectrum. Then we construct a Novikov resolution of $S$ as follows

$$\begin{array}{l} \vdots \\ S_\sigma \to S_\sigma \wedge bo \\ \vdots \\ S_1 \to S_1 \wedge bo \\ S \to S \wedge bo. \end{array}$$

We apply the $E_2$ of the Adams spectral sequence to this tower and get a spectral sequence which converges to $H^{**}(A)$ except for $s=t$. If we consider the resolution $X \wedge S_\sigma$, where $X$ is defined in Theorem 5, we can make an explicit calculation. Let

$$E_1^{s,t\sigma} = \mathrm{Ext}_A^{s-\sigma,t-\sigma}(\tilde{H}^*(X \wedge S_\sigma \wedge bo), Z_2).$$

PROPOSITION 8. *$E_2^{s,t,\sigma} = \sum_{\alpha \in P^\sigma} \mathrm{Ext}_A^{s-\alpha_t,t-\alpha_t}(M \otimes A//A_1, Z_2)$ for $s > \sigma$ where $P^\sigma$ is the set of $\sigma$-degree polynomials in the polynomial algebra introduced above.*

PROPOSITION 9. *$E_3^{s,t,\sigma} = E_\infty^{s,t,\sigma}$ for $s > \sigma$ and thus is given by Theorem* 3.

Note that Proposition 9 alone gives an edge of $3\sigma > t-2$. The sharpened version of Theorem 3 follows from Proposition 9 and a closer analysis of the nature of $\mathrm{Ext}_A^{s,t}(M, Z_2)$.

The most direct route from Proposition 8 to the main theorem requires a geometric realization of the $E_2$ term of the above spectral sequence for $S$. Using this resolution and the homotopy functor we get a spectral sequence whose $E_2^{\sigma,t}$ term has an edge of $5\sigma \geqq t-\sigma+\epsilon$. The image of $J$ has filtration 1. From this information the order of im $J$ should follow directly but no direct route has been found. Hence to complete the argument, consider the space $Y$ which is the fiber of the map $S \to K(Z, 0)$, and consider the resolution of $Y$ given by $\cdots \to Y \wedge S_\sigma \to Y \wedge S_{\sigma-1} \to \cdots$. Calculation of the sort given in the proof of III 7.3 of [4] and applied to elements of filtration zero and one give a proof of Theorems 1 and 2.

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