

L_p EMBEDDING AND NONLINEAR EIGENVALUE PROBLEMS FOR UNBOUNDED DOMAINS

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Let \mathbf{R}^N denote real N -dimensional Euclidean space. Then it is a well-known fact that the imbedding of the Sobolev space $W_{1,2}(\mathbf{R}^N)$ in $L_p(\mathbf{R}^N)$ is bounded for $2 \leq p \leq 2N/(N-2)$, but is definitely not compact. Consequently the theory of critical points for general isoperimetric variational problems defined over arbitrary unbounded domains in \mathbf{R}^N has been little investigated despite its importance. Indeed the usual proofs for the existence of even one critical point for such problems requires the verification of some compactness criterion (such as Condition C of Palais-Smale). For quadratic functionals, such as arise in the study of the spectrum of a linear elliptic partial differential operator L of order $2m$ defined on \mathbf{R}^N , many compact imbedding theorems have been obtained in recent years [1], [2], [3]. These results yield, in turn, interesting facts concerning the discrete spectrum of L . In this note we extend these compactness results to insure the existence of critical points for certain isoperimetric variational problems arising in the study of eigenvalue problems for nonlinear elliptic partial differential equations. The existence of stationary states for nonlinear wave equations [4] provides a natural example for which our results are useful.

1. Imbedding theorems. Throughout this note let Ω be an arbitrary (not necessarily bounded) domain in \mathbf{R}^N whose boundary $\partial\Omega$ is mildly smooth (say locally Lipschitzian). By $W_{s,p}(\Omega)$ we denote the Banach space of functions $u(x)$ defined on Ω such that u and all its partial derivatives up to and including order s are in $L_p(\Omega)$ (i.e. $D^\alpha u \in L_p(\Omega)$ for $|\alpha| \leq s$). The norm in $W_{s,p}(\Omega)$ (denoted by $\| \cdot \|_{s,p}$) is $\|u\|_{s,p} = \left\{ \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L_p}^p \right\}^{1/p}$. In order to state results on the imbedding of $W_{s,p}(\Omega)$ we introduce the functional $M_{\alpha,p,\Omega}(w)$ for any measurable function $w(x)$ defined on Ω , $\alpha + N > 0$, and $1 < p < \infty$ by setting

$$M_{\alpha,p,\Omega}(w) = \sup_x \int_{|x-y| < 1; y \in \Omega} |w(y)|^p \omega_\alpha(y) dy$$

where $\omega_\alpha(x) = |x|^\alpha$ for $\alpha < 0$ and $x \in \Omega$; 1 for $\alpha \geq 0$ and $x \in \Omega$.

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Actually we imbed $W_{s,p}(\Omega)$ in an $L_q(\Omega)$ space whose norm is weighted by the function $w(x)$ provided $M_{\alpha,p,\Omega}(w) < \infty$. This fact can be expressed as follows:

THEOREM 1. *Let Ω be any domain in R^N , and $w(x)$ a measurable function defined on Ω . Then the linear multiplication map $Lu = w \cdot u$ is a bounded linear map from $W_{s,p}(\Omega)$ to $L_q(\Omega)$ provided $s > 0$; $q \geq p > 1$; $1/q > 1/p - s/N$; $M_{\alpha,q,\Omega}(w) < \infty$ and $-N < \alpha < q(s - N/p)$. Furthermore*

$$\|w \cdot u\|_{0,q} \leq c[M_{\alpha,q,\Omega}(w)]^{1/q} \|u\|_{s,p}$$

where c is a constant depending only on p, q, s, N, Ω and α . Under the above hypotheses, the map L is compact provided

$$\int_{|x-y| < 1; y \in \Omega} |w(y)|^q dy \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

Moreover the above results hold if s is fractional as in [5].

As an illustration of this theorem, even if the origin is in Ω and $|x|$ denotes the distance from the origin, then we have

COROLLARY 2. *Let $s > 0$; $q \geq p > 1$ and suppose $0 < \beta < s + N(1/q - 1/p)$. Then the map $Lu = |x|^{-\beta} u$ is a bounded compact linear map from $W_{s,p}(\Omega)$ to $L_q(\Omega)$. Furthermore, there is a constant C , independent of u , such that*

$$\| |x|^{-\beta} u \|_{0,q} \leq C \|u\|_{s,p}.$$

REMARKS. 1. The number β of Corollary 2 is sharp in the sense that if $\beta \geq s + N(1/q - 1/p)$, then the map L is no longer compact, in general.

2. Theorem 1 and Corollary 2 apply as stated when $W_{s,p}(\Omega)$ is replaced by the usual $H_{s,p}(\Omega)$ spaces.

2. Critical point theory for isoperimetric variational problems. As a typical application of the previous compactness results, consider the following problem:

(P) Determine the critical points \mathfrak{S} of the functional $\mathfrak{A}(u) = \int_{\Omega} A(x, u, \dots, D^s u)$ subject to the constraint defined by $\mathfrak{B}(u) = \int_{\Omega} B(x, u) = \text{const}$, where Ω is an arbitrary domain in R^N .

Here we determine conditions on the functions A and B such that (i) the set \mathfrak{S} is nonvacuous and (ii) the critical point theories of Morse and Ljusternik-Schnirelmann are applicable to (P). To this end we suppose that for some positive integer s and some positive number p ($1 < p < \infty$), $\mathfrak{A}(u)$ is defined for all $u \in W_{s,p}(\Omega)$. We assume, for

simplicity, that A is a C^2 function of its arguments, and denote the derivative of $\mathfrak{A}(u)$ by $\mathfrak{A}'(u)$. Then we suppose:

- (a) $\mathfrak{A}(u) \rightarrow \infty$ as $\|u\|_{s,p} \rightarrow \infty$;
- (b) $\mathfrak{A}'(u)$ is a bounded, continuous map of $W_{s,p}(\Omega) \rightarrow W_{-s,p^*}(\Omega)$ (where $1/p + 1/p^* = 1$).
- (c) $\mathfrak{A}'(u)$ has the following closure property: if $u_n \rightarrow u$ weakly in $W_{s,p}(\Omega)$ and $\mathfrak{A}'(u_n) \rightarrow v$ strongly in $W_{-s,p^*}(\Omega)$, then $\mathfrak{A}'(u) = v$. (This holds if, for example, \mathfrak{A}' is a monotone map.)

Concerning the functional $\mathfrak{B}(u)$ we suppose

- (d) $|B(x, u)| \leq k(x)u^q$ where q and $k(x)$ are so chosen that the linear map $Lu = (k(x))^{1/q}u$ is a compact map of $W_{s,p}(\Omega) \rightarrow L_q(\Omega)$ (in accord with Theorem 1).

- (e) $B(x, u)$ is C^1 in u , $B_u(x, u)$ is Lipschitz continuous in u , and measurable in x for fixed u , with $uB_u(x, u) > 0$ for $u \neq 0$.

THEOREM 3. *Under the hypotheses (a)–(e) above, the functional $\mathfrak{A}(u)$ defined on the hypersurface $\mathfrak{M}_R = \{u \mid u \in W_{s,p}(\Omega), \mathfrak{B}(u) = R, R \text{ a positive constant}\}$ in $W_{s,p}(\Omega)$ satisfies the Palais-Smale Condition C, and consequently $\inf_{\mathfrak{M}_R} \mathfrak{A}(u)$ is a critical point of $\mathfrak{A}(u)$ on \mathfrak{M}_R .*

REMARK 3. An analogue of Theorem 3 holds for the conjugate variational problem (P*) (i.e. the critical point set of $\mathfrak{B}(u)$ on the hypersurface $\mathfrak{A}(u) = R$ in $W_{s,p}(\Omega)$) under appropriate coerciveness conditions on the form (u, Au) . As a simple example, consider

THEOREM 4. *Suppose A is a locally Lipschitz continuous bounded map from $W_{s,p}(\Omega) \rightarrow W_{-s,p^*}(\Omega)$ such that $(u - v, Au - Av) \geq c\|u - v\|^p$ for some constant $c > 0$ independent of $u, v \in W_{s,p}(\Omega)$. Then, if the functional $\mathfrak{B}(u)$ satisfies (d) and (e) above, $[B(u)]^{-1}$ defined on the hypersurface $\mathfrak{A}(u) = R$ in $W_{s,p}(\Omega)$ satisfies the Palais-Smale Condition C. Thus if $\mathfrak{A}(u)$ and $\mathfrak{B}(u)$ are even functionals, the operator equation $Au = \lambda Bu$ has a countably infinite number of distinct solutions (u_n, λ_n) with $\mathfrak{A}(u_n) = R$ and $\lambda_n \rightarrow \infty$.*

REMARK 4. Theorem 3 can be easily extended to hold for functionals $\mathfrak{B}(u) = \int_{\Omega} B(x, u, \dots, D^{s-1}u)$, since Theorem 1 can be generalized to the case of maps $L: W_{s,p}(\Omega) \rightarrow W_{t,q}(\Omega)$ with $s > t$ in a straightforward way.

Furthermore Theorems 3 and 4 hold when $W_{s,p}(\Omega)$ is replaced by appropriate closed subspaces so that the desired critical points satisfy null selfadjoint boundary conditions.

3. A nonlinear effect. The above results imply that the solvability of isoperimetric variational problems for unbounded domains re-

quires some decay at infinity for the functional $\mathfrak{B}(u)$. For many *non-linear* problems this decay need not appear explicitly. Indeed, consider the following simple special case (\tilde{P}) of the problem (P) with $\mathfrak{A}(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)$ and $\mathfrak{B}(u) = \int_{\mathbb{R}^N} F(u)$ ($N > 1$). To apply the results of §§1, 2 to this problem, the following transformation is useful. Restrict attention to functions $u(x)$ which depend only on $|x| = r$, and set $v(r) = r^{(N-1)/2}u(|x|)$. Then the appropriate functionals are

$$\tilde{\mathfrak{A}}(v) = \int_0^\infty \left[\dot{v}^2 + \left(1 + \frac{(N-3)(N-1)}{4r^2} \right) v^2 \right] dr$$

and

$$\tilde{\mathfrak{B}}(v) = \int_0^\infty F(r^{(1-N)/2}v) r(N-1) dr$$

over $\tilde{W}_{1,2}(0, \infty)$. Applying Corollary 2 and the Ljusternik-Schnirelmann theory of critical points, we find

THEOREM 5. *Suppose (i) $F'(y)$ is Lipschitz continuous and odd with $F'(y)y > 0$ and (ii) $|F(y)| \leq k|y|^\sigma$ for some $2 < \sigma < 2N/(N-2)$, then the critical point set \tilde{s} of P is nonempty, furthermore the critical point set \tilde{s}^* of the conjugate variational problem (\tilde{P}^*) contains a countably infinite number of distinct points. Consequently the equation $\Delta u - u + \lambda F'(u) = 0$ has a countably infinite number of distinct normalized radially symmetric solutions (u_n, λ_n) with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$.*

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