THE VITALI-HAHN-SAKS AND NIKODYM THEOREMS FOR ADDITIVE SET FUNCTIONS

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ABSTRACT. The purpose of this note is twofold. Firstly, we point out an appropriate version of the Vitali-Hahn-Saks and Nikodym theorems for finitely additive set functions defined on a sigma algebra of sets. Secondly, we apply our Theorem to extend a recent result of James K. Brooks for countably additive vector valued set functions to the general finitely additive case.

THEOREM. *Let* $\{\mu_n\}$ *be a sequence of bounded and finitely additive scalar valued set functions defined on a sigma algebra* Σ *of subsets of a set S. If* $\mu(E)=\lim_n \mu_n(E)$ *exists for every* $E\in\Sigma$*, then* μ *is bounded and additive and the additivity of the* μ_n *is uniform in n.*

In addition, suppose $\lim_{\nu(E)\to 0} \mu_n(E)=0$ for each n, where ν is a nonnegative finitely additive set function defined on Σ. Then $\lim_{\nu(E)\to 0} \mu_n(E)=0$ uniformly in n.

Our Theorem follows from the weak convergence theory for finitely additive set functions. While a proof of it can be synthesized (modulo an observation) from [3], [5], [6] and [7], for the readers' convenience we shall also refer to the discussion of the work of Soloman Leader and Pasquale Porcelli [6] and [7] given in [4]. The corollary on p. 475 of [3] tells us that the sequence $\{\mu_n\}$ is weakly convergent. Also, the (L)-space of bounded and additive functions on Σ is weakly complete [5, Theorem 12], so the sequence $\{\mu_n\}$ converges weakly to μ. Hence μ is bounded and additive, and (cf. [4]) the weakly convergent sequence $\{\mu_n\}$ is equi-absolutely continuous with respect to the bounded and additive function φ defined on Σ by

$$\varphi(E) = \sum_{k=1}^{\infty} 2^{-k}(1 + |\mu_k|(S))^{-1}|\mu_k|(E), \tag{1}$$

where $|\mu_k|$ is the variation of μ_k. Since the sequence $\{\mu_n-\mu\}$ converges weakly to zero, Lemma 1 of [4] implies

$$\lim_j\left\{\sup_k\left[\sum_{i\geq j}|\mu_k(E_i) - \mu(E_i)|\right]\right\} = 0 \tag{2}$$

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whenever $\{E_i\}$ is a sequence of pairwise disjoint elements of Σ. Moreover, because μ is bounded and additive, $\sum_{i\geq 1}|\mu(E_i)| \leqq |\mu| < \infty$. Hence

$$\lim_j \left\{ \sup_k \left[\sum_{i\geq j} | \mu_k(E_i) | \right] \right\} = 0 \tag{3}$$

whenever $\{E_i\}$ is a sequence of pairwise disjoint elements of Σ. In the countably additive case, uniform countable additivity is equivalent to (3). Thus (3) represents a reasonable definition of uniform additivity for the sequence $\{\mu_n\}$. Finally, if each μ_n is absolutely continuous with respect to ν, then φ is absolutely continuous with respect to ν.

Suppose that $\mathcal{B}$ is a separable Banach space over the complex numbers.

COROLLARY. *Let μ_n be a sequence of finitely additive $\mathcal{B}$-valued set functions defined on Σ such that* $\lim_n \mu_n(E)$ *exists for every* $E \in \Sigma$. *Suppose* $\lim_{\nu(E)\to 0} \mu_n(E) = 0$ *for each n, where ν is a nonnegative (real valued) finitely additive set function defined on Σ. Then* $\lim_{\nu(E)\to 0} \mu_n(E) = 0$ *uniformly in n.*

The proof given by Brooks in [1] for the countably additive version of this Corollary carries over if our Theorem is used and one notices that the μ_n of the Corollary is bounded and additive on Σ by Theorem 3.2 of [2].

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