THE K-SPAN OF A RIEMANN SURFACE

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Communicated by F. W. Gehring, May 11, 1970

In this note, we shall answer a question first posed by Sario and Oikawa in their monograph *Capacity functions* [4], and later by Rodin [3] in his Bulletin paper. The question is this. Does the class KD of harmonic functions u with finite Dirichlet integral and such that $*du$ has vanishing periods along all dividing cycles consist only of constant functions if and only if the K-span (for $m=0$) vanishes at some point with respect to some local parameter about it? The K-span (for $m=0$) is defined to be $\partial v/\partial x|_{z=\zeta}$ where dv reproduces for the space dKD, i.e. $(du, dv)=\pi\partial u/\partial x|_{z=\zeta}$ for all $du\in dKD$, and z denotes a local variable at ζ. Note that the definition of the span depends on ζ and the choice of the local variable at ζ.

We shall answer this question in the negative by exhibiting a Riemann surface which carries nonconstant KD functions but which has the property that the K-span (for $m=0$) vanishes at each point for some choice of the local variable at that point. Note that our K-span (for $m=0$) is Rodin's 1-span.

The Riemann surface we shall construct is the same one that appears on p. 377 of my paper *Boundaries of function spaces of Riemann surfaces* [2], but, in order to aid the reader, the details of the construction will be repeated here.

Let R_0 be a hyperbolic Riemann surface which admits no nonconstant harmonic functions with finite Dirichlet integral and has a single ideal boundary component. Let $\{\gamma_n\}$ denote a sequence of analytic Jordan arcs on R_0 such that $\gamma_n\cap\gamma_m=\emptyset$ for $n\neq m$, and such that for an arbitrary compact subset K of R_0, $\gamma_n\cap K=\emptyset$ for all sufficiently large n. Let $R'=R_0-\bigcup_{n=1}^{\infty}\gamma_n$ and take the sequence $\{\gamma_n\}$ such that R' does not belong to the class SO_{HD}, i.e. such that there exists a nonnegative Dirichlet function on R_0 which is harmonic on R' and vanishes quasi everywhere on $\bigcup_{n=1}^{\infty}\gamma_n$ but does not vanish quasi everywhere on R_0. Let R_1' and R_2' be two copies of R'. Denote by γ_n^+ (resp. γ_n^-) the positive (resp. negative) edge of γ_n. For each n, identify γ_n^+ of R_1' with γ_n^- of R_2' and γ_n^- of R_1' with γ_n^+ of R_2'. The resulting Riemann surface R has a single ideal boundary component. Furthermore, $R\in 0_{HD}^2-0_{HD}^1$, i.e. the dimension of the vector

AMS 1970 *subject classifications*. Primary 3045; Secondary 3111.

Key words and phrases. HD functions, KD functions, level curve, critical point, span, ideal boundary component.

lattice HD of harmonic functions on R with finite Dirichlet integral is 2. Since R has a single ideal boundary component, $KD=HD$, and hence R possesses nonconstant KD functions. Let ζ be a point of R and u a nonconstant KD function on R. Consider the level curve $u=u(\zeta)$. If ζ is a critical point of u, then since KD is two dimensional, ζ will be a critical point of every KD function and hence the K-span (for $m=0$) will vanish at ζ for every choice of the local variable at ζ. If ζ is not a critical point of u, then a portion of the level curve $u=u(\zeta)$ passing through ζ will be an analytic Jordan arc γ. Hence there exists a parametric disk (V, ϕ) about ζ such that $\phi(\gamma) = \{z: -1<x<1,\ y=0\}$. Since u is constant along $\phi(\gamma)$, it follows that $\partial u/\partial x$ vanishes at ζ. If dv denotes the reproducing kernel for the space dKD at ζ, the choice of the local variable at ζ will be that determined by ϕ, it follows that $\partial v/\partial x|_{z=\zeta}=0$, i.e. the K-span (for $m=0$) vanishes at ζ, since KD is two dimensional.

It is interesting to note that every KD function u takes the same value at each of the branch points. To see this, let $\sigma: R\to R$ denote the natural involution map of R onto itself, i.e. if ζ' and ζ'' lie over the same point ζ of R_0, then $\sigma(\zeta')=\sigma(\zeta'')$. Since R_0 admits no nonconstant HD functions, it follows that $u+u\circ\sigma=c$ where c is a constant. Hence if P is a branch point of R, $u(P)=c/2$ for all KD functions u.

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