SMOOTHINGS[1] AND HOMEOMORPHISMS FOR HILBERT MANIFOLDS

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0. The aim of this paper is to announce several results concerning smoothing Hilbert manifolds and some of their consequences for homeomorphisms. All manifolds are assumed to be Hausdorff, paracompact, and separable with local model the ∞-dimensional separable Hilbert space H. (All separable infinite dimensional Fréchet spaces are homeomorphic to H.) As usual, we denote by D^∞ the unit disc and by S^∞ the unit sphere in H. A pair $(M, \partial M)$ is called a *differentiable* manifold with boundary if it has a C^∞-differentiable structure with local model the Hilbert half space $H\times[0, \infty)$. The boundary ∂M, is the set of those points which are mapped onto $H\times\{0\}$ by some chart. It can be shown that ∂M is a differentiable manifold and is locally collared and thus collared in M. (See for instance, the existence of closed tubular neighborhoods [11].) This definition is not adequate for *topological* manifolds with boundary, because by Klee [10], $H\times[0, \infty)$ is homeomorphic to H. We thus define a topological manifold with boundary to be a pair $(M, \partial M)$ such that (a) M and ∂M are topological manifolds, (b) ∂M is collared in M, or equivalently locally collared (according to M. Brown, *Topology of 3-manifolds and related topics*, edited by M. K. Fort, p. 88). Our results are the following:

THEOREM 0.1. *Any homotopy equivalence of pairs* $f:(N, \partial N)\to(M, \partial M)$ *between differentiable (topological) manifolds with boundary is homotopic to a diffeomorphism (homeomorphism)* h *of pairs; moreover,* $h/\partial N$ *can be any diffeomorphism* $l:\partial N\to\partial M$ *homotopic in* ∂M *to* $f/\partial N$ *or any homeomorphism* $l:\partial N\to\partial M$ *homotopic in* M *to* $f/\partial N$.

In the differentiable case this theorem is a consequence of [3] or [7]. In the topological case, since M and N are topological manifolds with the same homotopy type, there exists a homeomor-

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[1] Smoothing, differentiable, diffeomorphism, etc., mean C^∞-smoothing, C^∞-differentiable, C^∞-diffeomorphism, etc.

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phism $g: N \to M$. Both ∂M and $g(\partial N)$ are Z-sets in M, and $g/\partial N$ is homotopic to l. Therefore, by the Theorem 1.1 below, there is a homeomorphism $k: N \to N$ such that $k \cdot g/\partial N = l$. Set $h = k \cdot g$.

REMARK. Theorem 0.1 in the topological case is true for nonseparable Hilbert manifolds, as follows from some results of Henderson [9] and Cutler (*Deficiency in F-manifolds*, to appear).

THEOREM 0.2. *Any topological manifold with boundary is homeomorphic to a differentiable manifold with boundary.*

This follows by combining Theorem 0.1 with Proposition 1.2 below.

THEOREM 0.3. *Given M, a differentiable manifold, and $h: M \to M$, a homeomorphism, then there exists a diffeomorphism $h': M \to M$ topologically isotopic to h.*

THEOREM 0.4. *Two homotopic homeomorphisms of a topological manifold M are topologically isotopic.*[3]

The theorem is an immediate consequence of Theorem 0.3 and [3, Theorem 2.3].

REMARK. Theorem 0.3 should be considered simultaneously with the following result of D. Henderson [8]: "Any topological manifold is homeomorphic to an open set of H, consequently to a differentiable manifold." The proofs of Theorems 0.3 and 0.4 are obtained using the handle decomposition [3] and will be sketched in §2. To be able to work with topological handles, in the case of topological manifolds, we make use of the uniqueness of collars by an ambient isotopy, Theorem 1.3, and in special cases the uniqueness of closed tubular neighborhoods by an ambient isotopy (Theorem 1.5). In order to isotop homeomorphisms into diffeomorphisms we smooth handles, make use of Theorem 1.6 and also move closed topological submanifolds with open interiors into differentiable ones by an ambient isotopy (Theorem 1.8); these theorems (1.3, 1.5, 1.6) use essentially Theorem 0.1 and also some other results listed in §1 as propositions.

1. THEOREM 1.1 [1]. *Let M and N be topological manifolds, K a closed subset of M and $h: M \to N$ a homeomorphism. Suppose $l: K \to N$ is a closed imbedding such that l is homotopic to h/K in N, and $l(K)$ and*

[3] h_0, h_1 two homeomorphisms (diffeomorphisms) of M are isotopic iff there exists a homeomorphism (diffeomorphism) $\mathcal{H}: M \times I \to M \times I$ such that $\mathcal{H}(x, t) = (\mathcal{H}_t(x), t)$ $\mathcal{H}_t(x) \in M$, $\mathcal{H}/M \times \{0\} = h_0$ and $\mathcal{H}/M \times \{1\} = h_1$. (Always I means $[0, 1]$.)

[4] A closed subset $K \subset M$ is called a Z-set in M iff for any nonempty homotopically trivial, open set U in M, $U \backslash K$ is a nonempty and homotopically trivial set in M.

$h(K)$ are ***Z**-sets*[4] *in N. Then there exists an isotopy of homeomorphisms $h_t: M \to N$, such that $h_0 = h$ and h_1 extends l.*

REMARK. This theorem has been extended in many cases to non-separable Fréchet manifolds (Henderson [9] and Cutler (thesis, Cornell University)).

PROPOSITION 1.2 [2]. *Given a pair of countable* CW-*complexes (A, B) $(B \subset A)$, there exists a differentiable manifold with boundary $(M, \partial M)$ and a homotopy equivalence of pairs $h: (A, B) \to (M, \partial M)$.*

THEOREM 1.3. *Let $(W, \partial W)$ be a topological manifold with boundary and let $\phi_i: \partial W \times [0, 1] \to W$, be two closed collar neighborhoods (this means $\phi_i | \partial W \times \{0\} = \mathrm{id}$, $V_i = \phi_i(\partial W \times [0, 1])$ are closed in W, and ϕ_i extend to open imbeddings $\bar{\phi}_i: \partial W \times [0, \infty) \to W$). Then for any open set, $U \supset V_1 \cup V_2$, there exists an isotopy of homeomorphisms $h_t: (W, \partial W) \to (W, \partial W)$ such that* (i) $h_0 = \mathrm{id}$, (ii) $h_t/(W \setminus U) \cup \partial W = \mathrm{id}$ *and* $h_1 \phi_1 = \phi_2$.

The proof is accomplished by first isotoping ϕ_1 to ϕ_1' so that $\phi_1'(\partial W \times [0, 1]) \subset \phi_2(\partial W \times [0, 1])$ and using Theorem 0.1 to show that $\phi_2(\partial W \times [0, 1]) \setminus \phi_1'(\partial W \times [0, 1])$ is an appropriate topological product.

REMARK. In the differentiable case we have a stronger analog of Theorem 1.3—see, for instance, the unicity of the closed tubular neighborhoods by an ambient isotopy [4].

PROPOSITION 1.4. *Let $h: (D^n \times D^\infty, \partial D^n \times D^\infty) \to (D^n \times D^\infty, \partial(D^n \times D^\infty))$ be a homeomorphism such that $h/\partial D^n \times D^\infty = \mathrm{id}$. There exists an isotopy of homeomorphisms $h_t: (D^n \times D^\infty, \partial(D^n \times D^\infty)) \to (D^n \times D^\infty, \partial(D^n \times D^\infty))$, $t \in [0, 1]$ such, that $h_0 = h$, h_t is a homeomorphism of pairs with $h_t/\partial(D^n \times D^\infty) = h$ and $h_1/L = \mathrm{id}$ (where $L = \{(x, y) \in D^n \times D^\infty \mid \|y\| \leq \frac{1}{2}(\|x\|^2 + 1)\}$).*

The proof uses Theorem 0.1 to obtain a homeomorphism g of $D^n \times D^\infty$ such that $g | L = \mathrm{id}$ and $g | D^n \times \partial D^\infty = h$, and then uses Alexander's trick which allow us to construct an isotopy

$$h_t(D^\infty, \partial D^\infty) \to (D^\infty, \partial D^\infty) h_t/\partial D^\infty = \mathrm{id}$$

between two homeomorphisms

$$h_i: (D^\infty, \ \partial D^\infty) \to (D^\infty, \partial D^\infty) h_i/\partial D^\infty = \mathrm{id}, \quad i = 0, 1.$$

THEOREM 1.5. *Let $(M, \partial M)$ be a topological manifold with boundary. Let $(D^n, \partial D^n) \to (M, \partial M)$ be a closed imbedding such that $i(D^n) \cap \partial M = i(\partial D^n)$, $Q' \subset Q$, two open neighborhoods of ∂D^n in D^n and $\phi_i: D^n \times D^\infty \to M$ two closed tubular neighborhoods (closed imbeddings which extend to the open imbeddings $\bar{\phi}_i: D^n \times H$ with $\bar{\phi}_i^{-1}(\partial M) = D^n \times H$). Suppose*

$\tilde{\phi}_1/Q\times H=\tilde{\phi}_2/Q\times H$. Then for any open neighborhood U of $\phi_1((D^n/Q')\times H)\cup\phi_2((D^n/Q')\times H)$ with $U\cap\partial M=\phi$, there exists an isotopy of homeomorphisms $\psi_t:(M, \partial M)\to(M, \partial M)$ such that (i) $\psi_0=\mathrm{id}$, (ii) $\psi_t/M\backslash U=\mathrm{id}$, (iii) $\psi_1\phi_1=\phi_2$.

REMARK. This theorem gives a weak version of the uniqueness of closed tubular neighborhoods by an ambient isotopy. In the differentiable case, a much stronger statement is proved in [4]. Theorem 1.5 is still true in the nonseparable cases using results of Henderson and Cutler. The theorem is reduced to Proposition 1.4 by using Theorem 0.1.

THEOREM 1.6. *Let $(K, \partial K)$ be a differentiable finite dimensional manifold with boundary, the disjoint union of at most a countable number of $(D^n, \partial D^n)$, and let $(M, \partial M)$ be a differentiable manifold with boundary. Let $Q'\subset\overline{Q}'\subset Q$ be open neighborhoods of ∂K, and let $\phi:(K\times D^\infty, \partial K\times D^\infty)\to(M, \partial M)$ be a topological closed imbedding which extends to an open imbedding $\tilde{\phi}:K\times H\to M$ which is differentiable on $Q\times H$ and for which $\tilde{\phi}^{-1}(\partial M)=\partial K\times H$. Then there exists an isotopy of homeomorphisms $h_t:(M, \partial M)\to(M, \partial M)$ such that* (i) $h_0=\mathrm{id}$, (ii) $h_t/M\backslash\phi(K\backslash Q')\times D^\infty)=\mathrm{id}$, (iii) *$h_1\cdot\phi/K$ is C^∞-differentiable.*

This theorem can be easily derived from Theorem 1.1 and the fact that $\phi/K\times\{0\}$ can be approximated by a C^∞-differentiable embedding $\tilde{\phi}:K\times\{0\}\to M$ such that $\tilde{\phi}/Q'\times\{0\}=\phi$ [3].

Let $(M, \partial M)$ be a topological, respectively, differentiable, manifold with boundary consisting of the disjoint union of two manifolds $\partial_0(M)$ and $\partial_1(M)$. A *topological*, (respectively, *differentiable*) *slicing submanifold* is a topological bicollared closed submanifold N, (respectively, a 1-codimensional closed submanifold N) such that: (i) $N\subset M\backslash\partial M$, (ii) N divides M in two manifolds with boundary $(M_1, \partial M_1)$, $(M_2, \partial M_2)$ where ∂M_1 is the disjoint union of $\partial_0 M$ and N and ∂M_2 is the disjoint union of N and $\partial_1 M$.

PROPOSITION 1.7. *Let $(M, \partial M)$ be a differentiable manifold and N a topological slicing submanifold. Then there exists a homeomorphism $h:(M, \partial M)\to(M, \partial M)$, $h(\partial M)=\mathrm{id}$, such that $h(N)$ is a differentiable slicing submanifold.*

THEOREM 1.8. *Suppose $(N, \partial N)$ is a closed topological submanifold with boundary of the differentiable manifold M such that ∂N is bicollared and $N\backslash\partial N$ is open. There exists a smooth collar neighborhood of $(N, \partial N)$, namely, a closed differentiably imbedded submanifold $(N', \partial N')$ with $N'\backslash\partial N'$ open, $N\subset N'\backslash\partial N'$, and $(N'\backslash\mathrm{Int}\,(N), \partial N'\cup\partial N)$ homeomorphic to $(\partial N\times[0, 1], \partial N\times(\{0\}\cup\{1\}))$ as pairs.*

The proof uses the Proposition 1.7, and also the result of [5] (§9 corollary).

2. We will make some comments concerning the proof of Theorems 0.3 and 0.4. To prove Theorem 0.3, first choose a handle decomposition $(A_i, \partial A_i)$ of M. This always exists according to [3] Proposition 1.2 and [8], [6] and [12]. Inductively, suppose we have a homeomorphism $h_n: M\to M$ and an isotopy of homeomorphisms $h_t: M\to M$, $t\in[n-1, n]$, such that $(\mathrm{i}_n)h_n$ restricted to a collar neighborhood of A_n is a diffeomorphism, $(\mathrm{ii}_n)h(A_{n+1})\supset(A_n)\supset h(A_n)$ and $(\mathrm{iii}_n)h_t|A_{n-1}=h_{n-1}$. We construct an isotopy h_t, $t\in[n+1, n]$, satisfying (i_{n+1}) and (iii_{n+1}) by applying Theorems 1.5 and 1.6. The isotopy can be improved to also satisfy (i_{n+1}) by applying Theorem 1.8 and the unicity of collar neighborhoods in the differentiable case ([3], Remarks (1) and (2)).

Theorem 0.4 follows according to [3 Theorem 2.3].

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