THE NONFINITE TYPE OF SOME $\text{Diff}_0 M^n$

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1. **Introduction.** Throughout this paper, unless explicitly stated otherwise, all manifolds will be closed, connected, oriented, and of class C^∞. We denote by X some arbitrary closed subset of the n-manifold M^n, $X \neq M^n$. When X is empty, we shall suppress it from the notation. The set of orientation-preserving self-diffeomorphisms of M^n that fix each point of X can be made into a locally-path-connected, metrizable, topological group $\text{Diff}(M^n, X)$ by endowing it with the usual C^∞ topology and the operation of map-composition [6], [8]. Let $\text{Diff}_0(M^n, X)$ be the identity component of $\text{Diff}(M^n, X)$. For general M^n, the only global homotopy-theoretic fact known about $\text{Diff}_0(M^n, X)$ is that it has the homotopy type of a countable CW complex [8].

Let S^n be the standard, oriented n-sphere. In [1], the authors announced that $\text{Diff}_0 S^n$ does not have the homotopy type of a finite CW complex when $n \geqq 7$. The techniques described in §3 of this announcement, together with [1], allow us to extend this result to other manifolds.

2. **Statement of the main results and remarks.** Our main result is the following:

2.1. THEOREM. *If M^n is a spin manifold with trivial rational Pontrjagin classes, then* $\text{Diff}_0(M^n, X)$ *does not have the homotopy type of a finite* CW *complex when either* (a) $n = 8k-4$, $k \geqq 6$, *or* (b) $n = 8k$ *and k is admissible.*

An *admissible natural number k* is a natural number $\geqq 42$ for which the open interval $(\frac{1}{3}(2k+1), \frac{1}{2}(2k+1))$ contains at least one prime. It follows from the Prime Number Theorem that there are at most finitely many *in*admissible natural numbers, but the precise value of the largest such number is not known. See Remark 2.5 below.

2.2. REMARK. It is clear that π-manifolds of the appropriate dimensions satisfy the hypotheses of 2.1. Note that this includes homotopy spheres, homotopy tori, real Stiefel manifolds, compact Lie

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groups, and nilmanifolds.

2.3. REMARK. J. Eells has conjectured that the standard n-torus T^n is a deformation retract of $\mathrm{Diff}_0 T^n$. This is true when $n=2$, [7]. The above theorem, however, shows that it is false, in general. Moreover, application of results of this announcement, together with those of [1], implies that the conjecture is false for all $n \geqq 25$. In a subsequent paper, whose techniques differ substantially from those described here, the authors improve this to $n \geqq 5$ (cf. Remark 3.2 (c)).

2.4. REMARK. Every closed, oriented surface M^2 admits a spin structure and has trivial Pontrjagin classes. However, results of Smale [9] and Earle and Eells [7] imply that $\mathrm{Diff}_0 M^2$ has finite type.

2.5. REMARK. Our proof of Theorem 2.1 combines the detection of nonzero $\pi_i(\mathrm{Diff}_0 M^n)$ (see 3.5 below) with a theorem of W. Browder [3] which implies that an arc-connected H-space Y of finite type satisfies $\pi_2(Y)=0$. Our method for detecting nonzero $\pi_i(\mathrm{Diff}_0 M^n)$, here, ultimately involves obtaining some crude lower bound for the order of a certain finite abelian group (see 3.9 below, and [1, Proposition 3.4]). This, in turn, involves certain elementary number-theoretic considerations which motivate our definition of *admissible natural number*, above.

2.6. COROLLARY. *Let M^n be a riemannian n-manifold with constant sectional curvature. If M^n admits a spin structure and either* (a) $n=8k-4$, $k \geqq 6$, *or* (b) $n=8k$, *k admissible, then* $\mathrm{Diff}_0(M^n, X)$ *does not have finite type.*

The following proposition shows that 2.6 is nonvacuous.

2.7. PROPOSITION. (a) *If M^n is a riemannian manifold of constant negative curvature, then there exists a finite riemannian covering $\tilde{M}^n \to M^n$ with $\tilde{M}^n$ a spin manifold.*

(b) *Suppose that G is a group of odd order $q>1$ and is generated by r elements. Let n be any integer $\geqq 3$ such that $n-1\equiv 0$* (mod q) *and $(n-1)/q \geqq r-1$. Then, there exists a flat riemannian n-manifold having G as a linear holonomy group and admitting a spin structure.*

We are indebted to Professor A. Borel for a proof of (a), above. Part (b) is an easy addendum to a theorem of Auslander and Kuranishi [2]. Note that the only even-dimensional (oriented!) manifolds of constant positive curvature are the standard spheres.

3. Detecting nonzero $\pi_i(\mathrm{Diff}_0 M^n)$.

3.1. *The groups* $\pi_i(\mathfrak{D}\mathrm{iff};\, M^n, X)$.

Let $\boldsymbol{R}^i$ denote Euclidean i-space. A $\mathfrak{D}\mathrm{iff}$ *map on* $M^n \times \boldsymbol{R}^i$ *rel X* is an orientation-preserving, C^∞ diffeomorphism $f: M^n \times \boldsymbol{R}^i \to M^n \times \boldsymbol{R}^i$

such that supp f (=closure$\{(p, y) \in M^n \times R^i \mid f(p, y) \neq (p, y)\}$) is compact and does not meet $X \times R^i$. A $\mathfrak{Diff}$-*concordance on* $M^n \times R^i$ *rel* X is an orientation-preserving C^∞ diffeomorphism $F: M^n \times R^i \times [0, 1] \to M^n \times R^i \times [0, 1]$ such that:

(i) supp F is compact and does not meet $X \times R^i \times [0, 1]$;

(ii) for some $\epsilon > 0$, F has the form $F(p, y, t) = (F_\alpha(p, y), t)$, if $|t - \alpha| \leqq \epsilon$, $\alpha = 0, 1$.

$\mathfrak{Diff}$-*concordance rel* X between $\mathfrak{Diff}$-maps on $M^n \times R^i$ rel X is now defined in the obvious way and is easily seen to be an equivalence relation. The set of equivalence classes becomes a group under composition of maps, abelian when $i \geqq 1$, and we denote it by $\pi_i(\mathfrak{Diff}; M^n, X)$.

3.2. REMARKS. (a) Note that every $\mathfrak{Diff}$-map on $M^n \times R^i$ rel X is required to fix each point of some neighborhood of $X \times R^i$, the neighborhood depending on the map. Let

$$\text{(1)} \qquad \mathrm{Diff}(M^n, N(X)) \subseteq \mathrm{Diff}(M^n, X)$$

be the subgroup consisting of all diffeomorphisms that fix some neighborhood of X, the neighborhood depending on the diffeomorphism. Clearly there is a forgetful homomorphism

$$\text{(2)} \qquad \pi_i(\mathrm{Diff}(M^n, N(X))) \to \pi_i(\mathfrak{Diff}; M^n, X).$$

Now, using results of Cerf [6], we can show that the inclusion (1) is a homotopy equivalence when X is a (possibly empty) compact, codimension-zero, submanifold-with-boundary. By means of this equivalence, therefore, we may replace (2) by the "forgetful" homomorphism

$$\Phi: \pi_i(\mathrm{Diff}(M^n, X)) \to \pi_i(\mathfrak{Diff}; M^n, X).$$

(b) Let $D^n_+ = \{x \in S^n \mid x_1 \geqq 0\}$. Recall that Γ^{n+i+1} is the Kervaire-Milnor group and that Γ^{n+i+1}_{i+1} is the Gromoll subgroup described in [1]. We can construct an isomorphism $\pi_i(\mathfrak{Diff}; S^n, D^n_+) \approx \Gamma^{n+i+1}$ under which $\Phi(\pi_i(\mathrm{Diff}(S^n, D^n_+)))$ is taken onto Γ^{n+i+1}_{i+1}.

(c) Analogues of $\pi_i(\mathfrak{Diff}; M^n, X)$ can be defined in other categories. The authors use these analogues to develop other methods for detecting nonzero $\pi_i(\mathrm{Diff}_0 M^n)$.

3.3. PROPOSITION. *There exist homomorphisms E_* and $\mathcal{E}_*$ making the following diagram commute*:

$$\begin{array}{ccc} \pi_i(\mathrm{Diff}(S^n, D^n_+)) & \xrightarrow{E_*} & \pi_i(\mathrm{Diff}\ M^n) \\ \Phi \downarrow & & \downarrow \Phi \\ \pi_i(\mathfrak{Diff}; S^n, D^n_+) & \xrightarrow{\mathcal{E}_*} & \pi_i(\mathfrak{Diff}; M^n) \end{array}$$

The definitions of E_* and $\mathcal{E}_*$ are similar. We define $\mathcal{E}_*$. Choose orientation-preserving C^∞-imbeddings $f: D^n \to S^n$ and $g: D^n \to M^n$, where D^n is the unit ball in $\mathbf{R}^n$, such that $f(\frac{1}{2}D^n) = D^n_-$. Given any $\mathfrak{Diff}$-map d on $S^n \times \mathbf{R}^i$ rel D^n_+, then

$$(g \times \mathrm{id}_{\mathbf{R}^i})(f^{-1} \times \mathrm{id}_{\mathbf{R}^i}) d (f \times \mathrm{id}_{\mathbf{R}^i})(g^{-1} \times \mathrm{id}_{\mathbf{R}^i})$$

extends uniquely to a $\mathfrak{Diff}$-map $\bar{d}$ on $M^n \times \mathbf{R}^i$ rel (M^n-interior $g(\frac{1}{2}D^n)$). We may consider $\bar{d}$ to be a $\mathfrak{Diff}$-map on $M^n \times \mathbf{R}^i$. The association $d \to \bar{d}$ defines $\mathcal{E}_*$. It is easy to show that E_* and $\mathcal{E}_*$ do not depend on the choice of f or g.

3.4. *A pasting construction.* Let $\mathfrak{M}^{n+i+1}$ be the set of C^∞, oriented-diffeomorphism-types of oriented, C^∞, $(n+i+1)$-manifolds. Let $\eta: \mathbf{R}^i \to S^i$ be a fixed, orientation-preserving diffeomorphism onto S^i minus a point, and let f be a $\mathfrak{Diff}$-map on $M^n \times \mathbf{R}^i$. Define $\bar{f}$ to be the compactification of $(\mathrm{id}_{M^n} \times \eta) f (\mathrm{id}_{M^n} \times \eta^{-1})$, and form the C^∞ oriented $(n+i+1)$-manifold $W(f) = M^n \times D^{i+1} \cup_{\bar{f}} M^n \times D^{i+1}$ by the standard pasting process. The association $f \to W(f)$ well defines a *function* $\pi_i(\mathfrak{Diff};\, M^n) \xrightarrow{P} \mathfrak{M}^{n+i+1}$.

Now recall that, by Remark 3.2 (b), above, each element $\phi \in \pi_i(\mathfrak{Diff};\, S^n,\, D^n_+)$ determines an oriented, homotopy $(n+i+1)$-sphere, unique up to orientation-preserving diffeomorphism, which we denote by $\Sigma(\phi)$. Let $\#$ denote the operation of connected sum.

3.5. PROPOSITION. *For every* $\phi \in \pi_i(\mathfrak{Diff};\, S^n,\, D^n_+)$,

$$P \circ \mathcal{E}_*(\phi) = [(M^n \times S^{i+1}) \# \Sigma(\phi)] \in \mathfrak{M}^{n+i+1}.$$

Recall that the *inertia subgroup* $I(M^n \times S^{i+1}) \subseteq \Gamma^{n+i+1}$ consists of all classes represented by oriented homotopy spheres Σ for which $(M^n \times S^{i+1}) \# \Sigma$ is orientation-preserving diffeomorphic to $M^n \times S^{i+1}$.

3.6. COROLLARY. *We identify* $\pi_i(\mathfrak{Diff};\, S^n,\, D^n_+)$ *and* Γ^{n+i+1} *by the isomorphism mentioned in Remark* 3.2 (b), *above. Then,*

$$\text{kernel } \mathcal{E}_* \subseteq I(M^n \times S^{i+1}).$$

Combining 3.2 (b) and 3.3 with 3.6, we obtain the following

3.7. NONTRIVIALITY CRITERION. *The homomorphism*

$$\pi_i(\mathrm{Diff}(S^n,\, D^n_+)) \xrightarrow{E_*} \pi_i(\mathrm{Diff}\, M^n)$$

is nontrivial provided that $\Gamma^{n+i+1}_{i+1} \not\subseteq I(M^n \times S^{i+1})$.

To apply 3.7, we prove the following:

3.8. THEOREM. *Let M^n be a spin manifold with vanishing rational Pontrjagin classes. Then,*

$$I(M^n \times S^{i+1}) \cap bP_{n+i+2} = 0, \qquad \text{if } n+i+1 \equiv 3\,(8), \quad i \geqq 1,$$
$$= Z_2 \quad \text{or} \quad 0, \qquad \text{if } n+i+1 \equiv 7\,(8), \quad i \geqq 1.$$

Here, $bP_{k+1} \subseteq \Gamma^k$ consists of all classes whose representatives bound π-manifolds.

This theorem and its proof are closely related to the work of W. Browder [4], with the important difference that his results require $H^1(M^n \times S^{i+1}) = 0$, whereas ours do not. For the proof, given a Σ representing a class in $I(M^n \times S^{i+1}) \cap bP_{n+i+2}$, we construct a manifold W such that:

(i) W is a spin manifold with trivial decomposable Pontrjagin numbers;

(ii) ∂W is orientation-preserving diffeomorphic to Σ;

(iii) Index $W=0$.

It follows that the Brumfiel invariant $\lambda(\Sigma)$ is zero, so that, by results of Brumfiel [5], Σ or $\Sigma \# \Sigma$ is standard, as stated above.

3.9. PROOF OF THEOREM 2.1. Combining 3.7 and 3.8, we can detect nonzero $\pi_i(\text{Diff } M^n)$ whenever either $\Gamma_{i+1}^{n+i+1} \cap bP_{n+i+2} \neq 0$, $n+i+1 \equiv 3$ (8), $i \geqq 1$, or $\Gamma_{i+1}^{n+i+1} \cap bP_{n+i+2}$ has more than two elements, $n+i+1 \equiv 7$ (8), $i \geqq 1$. Set $i=2$ and apply Propositions 3.3 and 3.4 of [1]. It follows (after some complicated but elementary number theory), that $\pi_2(\text{Diff } M^n) \neq 0$ when n satisfies 2.1 (a) or (b). It is not hard to show that E_* factors through $\pi_i(\text{Diff}(M^n, X))$, so that the same nontriviality result applies to $\pi_2(\text{Diff}(M^n, X))$.

Theorem 2.1 now follows by applying the theorem of Browder described in Remark 2.5. Q.E.D.

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