Singer. The Hoffman-Singer theory of maximal algebras and the Helson-Lowdenslager work on invariant subspaces and cocylces is made available for the first time in book form. Also included are extensions of the works of these authors due to deLeeuw and Glicksberg and Gamelin.

This review would be incomplete without a few words on the merits of these books as textbooks. Browder's book seems ideal for a one-semester course for students who already know some function theory and basic Banach space theory. Because of the more detailed treatment, the student may find it easier to read Browder. The first two chapters of Gamelin can be also used as material for an introductory course. The later chapters offer a magnificent selection of topics that can be offered in specialized courses. Browder's book does not offer any exercises. It also lacks a terminological index. Gamelin's book contains exercises of varying degrees of complexity. Some of the problems are actually theorems from recent papers. In such cases the author purposely omits the references. The reviewer feels that it would have been nicer to give references to some of the more difficult exercises.

M. RABINDRANATHAN

Mordell, Diophantine equations, by L. J. Mordell, Academic Press, New York and London, 1969.

The theory of diophantine equations is one of the oldest in mathematics, one of its most attractive, and also at the moment one which is still fairly undeveloped as being exceptionally hard. One reason for this is perhaps that in the full generality of the Hilbert problem, it cannot be effectively dealt with. Nevertheless, I personally would expect a wide class of diophantine problems to be effectively solvable (e.g. those on curves or abelian varieties), and in any case, many special cases are solvable.

Because of difficulties which have been encountered historically, a portion of the subject has developed as an accumulation of special diophantine equations, mostly in two variables, i.e. curves. It was well understood in the nineteenth century that nonsingular cubic curves have a group law on them, parametrized by the elliptic functions from a complex torus, but Poincaré was the first to draw attention to the special group of rational points when this curve is defined by an equation with rational coefficients, and he guessed that this group might be finitely generated. Mordell proved this fact in 1922, and thereby provided the first opportunity to behold the beginnings of a much broader approach to this type of equation. He also conjectured that a curve of genus $\geqq 2$ has only a finite number of rational points, and this magnificent conjecture remains unproved today. These matters, which are perhaps Mordell's greatest contributions to the subject, are treated in Chapters 16 and 17 of the present book.

The other parts of the book are roughly distributed as follows. A number of concrete special equations of degrees 2, 3 and 4 are discussed at the beginning, mostly with the method of congruences. Chapter 7 gives a discussion of the fundamental theorem concerning quadratic forms over the rationals (solvability globally is equivalent to solvability locally everywhere).

Chapter 8 deals with Pell's equation, which essentially solves effectively for the units of a real quadratic field. The treatment is classical. Next comes a sequence of chapters on surfaces, mostly cubic and quartic, dealing with special cases when rational or integral points can be found. A brief chapter mentions the role of units as affecting certain equations in number fields, and examples are worked out. After the general discussion already mentioned on curves of genus 1 or >1, we return to special cases which can be handled without the general theory, somewhat more effectively using Minkowski's theorem on convex bodies and congruence methods, applied to the representation of numbers by quadratic and cubic forms. Next we have Thue's theorem on diophantine approximations (the weak version, not Roth's version), and its application to equations of type $f(x, y)=m$ where f is a homogeneous polynomial of higher degree. The next chapter mentions Skolem's method by p-adic analysis, but does not go into details of proofs.

We then return to cubic and quartic forms, or rather special cases, involving the explicit determination of integral solutions. The discriminant and other covariants of these forms are discussed (indispensable means to get at the solutions effectively). The scene shifts back once more to cubic and quartic curves of the elliptic type, like $y^2=x^3+k$ and $y^2=f(x)$, where f is a cubic polynomial with no multiple roots, looking for integral points rather than rational points. Mordell's original proof that the number of these is finite is given, using Thue's theorem. The next chapter indicates the extension of this result to the case when f has arbitrary degree (proved by Siegel in 1926), but refers to an earlier Siegel paper for the stronger version of the analogue of Thue's theorem needed to make the proof go through. Mordell also states Siegel's general theorem of 1929, that a curve of genus at least 1 has only a finite number of integral points, giving explicitly the exceptional cases of genus 0 when infinitely many such points may occur. The book concludes with other special equations of higher degree, for instance special results on $ax^n - by^n = c$ and the Fermat curve, proving the nonexistence of integral solutions in a few simple cases, while assuming assorted facts of algebraic number theory, both of the standard variety, but also more specialized, like those involving regular primes.

As can be seen from this sketch, the contents of the book are jumpy, and some comments are now in order concerning the broader implications of Mordell's style, his point of view, and the context in which he writes.

Special concrete cases like cubic curves have provided much of the testing ground for experimentation, methods, theorems, and conjectures in diophantine analysis, and hence it is very welcome to have some of these cases brought together, as Mordell has done. I emphasize: He collects together special cases, without particular unifying order, or any design that I could make out, that might tie them together or make their succession in 30 chapters more than what appears to be an arbitrary succession. That is Mordell's taste, and I cannot quarrel with it. The book, as it is, will be very useful to those interested in diophantine equations, and wishing to work out special cases with essentially elementary techniques. I personally had bought a copy of the book before being sent the review copy, now given to the library.

But the reader must be aware of the limitations of Mordell's exposition. For one thing, Mordell clings systematically to the chronological development of the subject throughout the book. Even when an important development has taken place, e.g. Roth's theorem on diophantine approximations, subsuming previous results in the subject (by Thue and Siegel, say), Mordell gives the earliest theorem, namely Thue's, and only briefly refers to Roth's paper for the extension, when only a few additional pages at an equally elementary level would have been needed to get the full result. When a stronger version is needed to handle the equation $y^2 = f(x)$ with f of higher degree, Mordell refers to an earlier paper of Siegel (Math. Zeitschrift, 1921, a misprint in the reference gives the date erroneously as 1961), which also treats the number field case needed for this particular application. However, the inexperienced reader will have to figure out for himself that a single formulation of Roth's theorem in number fields can be used effectively for all these applications.

Even though I find the succession of equations treated somewhat arbitrary, there seems to be one thread which runs through them, suggested by the "List of Equations and Congruences" appearing at the end of the book in lieu of an index. This list is ordered according to degree (degree 1, degree 2, degree 3, degree 4, degree>4) and then according as to whether the equation is homogeneous or not. Of course, one's first attempt in dealing with diophantine equations is to experiment with equations of low degree and small coefficients. But it soon becomes apparent that the degree is not a good invariant for the behavior of these equations, whether searching for rational points or integral points, and the classification by degree is to a large extent misleading. However, Mordell's taste when faced with a theorem like Siegel's on curves of higher genus is just to say: "The proof is of a very advanced character." And leave it at that.

Nor does Mordell tell us of Weil's generalization in 1929 concerning the finite generation of the group of rational points in the higher dimensional case; which is a pity, because this is one of the approaches which gives a method of attack for the Mordell conjecture on curves of genus $\geqq 2$: We embed them in their Jacobians, and look at their intersection with the finitely generated group of rational points on this Jacobian. By this method, and the positive definite quadratic form of Néron-Tate on this group, Mumford was for instance able to show that the gaps between rational points of ascending height become exponentially large.

It is also possible to connect both results and methods of diophantine analysis with algebraic geometry, and I found it interesting in my book *Diophantine Geometry* to present the known results which allowed us to make this connection coherently (e.g. as it applies to Severi's theorem of the base, following work of Néron, and with Néron). The intense dislike which Mordell has for this kind of exposition is clearly evidenced by his famous review of the book (Bull. Amer. Math. Soc. **70** (1964), 491–498). (If the review is not famous, it should be.) In this connection, I can do no better than to reproduce an exchange of letters with him in November 1966, shortly after I had sent him a copy of some of my other books. He kindly wrote me:

> Dear Professor Lang, Thank you very much for the textbooks which I shall be glad to read. I hope I shall not have to struggle with them as I did with D. G. You may be interested to know that I found your *Algebra* quite readable and very useful. It was obviously meant to be understood. (You may quote this if you wish to.) . . . "

And after a few other kind remarks, he expressed the hope to meet me at a talk of his on diophantine equations to be given shortly in New York. (We met, and I enjoyed it.) Still, I answered Mordell on the substantial points raised both by his review and his letter:

Dear Professor Mordell, Thanks for your letter. What you write there prompts me to clarify some points about book writing.

I see no reason why it should be prohibited to write very advanced monographs, presupposing substantial knowledge in some fields, and thus allowing certain expositions at a level which may be appreciated only by a few, but achieves a certain coherence which would not otherwise be possible.

This of course does not preclude the writing of elementary monographs. For instance, I could rewrite Diophantine Geometry by working entirely on elliptic curves, and thus make the book understandable to any first year graduate student (not mentioning you $\cdots$). Both books would then coexist amicably, and neither would be better than the other. Each would achieve different ends.

When you write of any book that it is "obviously meant to be understood", whether as a compliment for one book or blame for another, you are still missing the point: I never meant Diophantine Geometry to be understood specifically by you, or anyone who did not have the rather vast background required for its reading. All my books are meant to be understood by readers having the prerequisites for the level at which the books are written. These prerequisites vary from book to book, depending on the subject matter, my mood, and other aesthetic feelings which I have at the moment of writing. When I write a standard text in Algebra, I attempt something very different from writing a book which for the first time gives a systematic point of view on the relations of diophantine equations and the advanced contexts of algebraic geometry. The purpose of the latter *is* to jazz things up as much as possible. The purpose of the former is to educate someone in the first steps which might eventually culminate in his knowing the jazz too, if his tastes allow him that path. And if his tastes don't, then my blessings to him also. This is known as aesthetic tolerance. But just as a composer of music (be it Bach or the Beatles), I have to take my responsibility as to what *I* consider to be beautiful, and write my books accordingly, not just with the intent of pleasing one segment of the population. Let pleasure then fall where it may.

With best regards,

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