BOOK REVIEWS

Introduction to function algebras by A. Browder. W. A. Benjamin, Inc., New York, 1969. 12+273 pp. $15.00.

Uniform algebras by T. W. Gamelin. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1969. 13+257 pp. $11.00.

Algèbres de fonctions et espaces de Hardy. par G. Lumer. Lecture Notes in Mathematics, No. 75, Springer-Verlag, Berlin, 1968. 3+80 pp. $2.20.

The three books under review are devoted to the study of uniform algebras and Hardy spaces associated with them. Naturally, their intersection is nonempty. However they are written for different purposes. Browder's book is an elaboration of the Lecture notes of a one semester course he taught. His aim is to give a self-contained treatment of some of the basic results of the subject. Gamelin's aim is to give a comprehensive survey of the subject. His exposition is not as detailed as Browder's. The abundance of the material included in Gamelin's book justifies this style to a great extent. The prerequisites needed to read Gamelin's book are more elaborate too. The main feature of Lumer's notes is his theory of Hardy spaces relative to a system of measures.

The idea of a uniform algebra (alias function algebra, sup norm algebra) is motivated by algebras of analytic functions on compact subsets of complex Euclidean spaces and by algebras of generalized analytic functions on certain compact groups. The term uniform algebra was coined by E. Bishop. Although uniform algebras form a rather special class of commutative Banach algebras, familiarity with Gelfand's theory of commutative Banach algebras is essential to understand uniform algebra theory. Both Browder and Gamelin begin with discussions of commutative Banach algebra theory. Gelfand theory and the functional calculus in one variable are treated in detail and various applications are given. To illuminate the general theory, maximal ideal spaces of various examples are identified. This task can sometimes be quite nontrivial as testified by a theorem of Arens. Another topic in Banach algebras that is studied is Gleason's embedding theorem. Browder has also given a discussion of point derivations while Gamelin includes that topic in an exercise.

A result of paramount importance in the theory of uniform algebras is the observation of Arens and Singer that a complex homomorphism of such an algebra may be represented by a probability measure in the space on which the algebra lives. All the books under review prove this fact and study various types of representing measures. Some familiar results of classical function theory receive abstract formulations via representing measures. The problem of finding representing measures with smallest supports is tied up with the existence problem of boundaries for uniform algebras. In proving the existence of the Šilov boundary, Browder follows the Choquet theoretic approach. Gamelin gives the recent elegant proof due to Hörmander.

In Chapter 3 Gamelin considers the extension of the functional calculus to several variables. This chapter gives a thorough discussion of the applications of the techniques of several complex variables to commutative Banach algebras. He follows the approach of Waelbroeck in the construction of functional calculus. This approach has the advantage that the Oka-Weil theorem on polynomial approximation is not used in the construction of the functional calculus. In fact the approximation theorem is deduced from the functional calculus. But one cannot get away without using some theorems from several complex variables. In the proof of the Arens-Royden theorem on the cohomology of maximal ideal spaces, the solvability of the first Cousin problem for polynomial polyhedra is used. Also used is a result of Bruschlinsky from cohomology theory. (Unfortunately, no reference is given for this result.) Among other applications of several complex variables are Šilov's idempotent theorem and Rossi's local maximum principle.

Many problems in approximation theory may be formulated as questions of determining whether a certain uniform algebra is equal to a uniform algebra contained in it. A functional analyst's recipe for the solution would be to show that any measure that annihilates the smaller algebra annihilates the larger one. This approach has met with remarkable success in giving functional analytic proofs of well-known approximation theorems like those of Runge and Mergeljan. All the books under review give proofs of Mergeljan's theorem. The proof can be conveniently divided into two parts. The first part is a classical approximation theorem of Lebesgue and Walsh. Lumer assumes this result in his exposition. Gamelin's treatment is based on Walsh's original paper. In this approach knowledge of Dirichlet's principle is assumed. Browder gives an approach which is based on the ideas of Lennart Carleson. This approach is indeed very elegant and completely elementary. The second part of the functional analytic proof of Mergeljan's theorem deals with the structure of measures that annihilate algebras of rational functions. This problem can actually be studied for an abstract uniform algebra. The goal is to decompose an annihilating measure into a sum of annihilating measures, one of which is singular with respect to all representing measures; each of the others being absolutely continuous with respect to some representing measure. This program can be carried out in varying degrees of generality. For proving Mergeljan's polynomial approximation theorem, it suffices to prove the decomposition theorem when the underlying algebra is a Dirichlet algebra. This is the original form of the decomposition theorem due to Glicksberg and Wermer. Browder gives essentially this version. The basic tool for this development is a generalization of a classical theorem of the Riesz brothers. As Glicksberg showed later, the decomposition theorem can be proved without any density conditions on the algebra. This most general form of the theorem is proved by both Gamelin and Lumer. As we mentioned earlier, in the decomposition of an annihilating measure there is a summand that is singular with respect to all representing measures. Such a measure can cause a lot of headache. Fortunately, for the algebra that one has to deal with in proving Mergeljan's theorem, such measures do not arise. This useful result of Wilken is proved with the aid of the Cauchy transform. It was E. Bishop who showed how the Cauchy transform can be exploited very effectively in connection with rational approximation problems. The books under review contain numerous applications of the Cauchy transform, many of them due to Bishop.

In the final chapter, Gamelin takes up the basic rational approximation problem again. This chapter contains proofs of recent theorems of Vituškin and Melnikov. Now the techniques used are more function theoretic and more constructive. As Gamelin remarks, the connection between these constructive techniques and the abstract method is not completely understood.

An interesting topic in the topology of maximal ideal spaces is the question whether a maximal ideal space or parts of it can be endowed with an analytic structure which is compatible with the given uniform algebra. One such theorem is Gleason's embedding theorem mentioned earlier. A landmark in this theory is Wermer's embedding theorem. A concept which arises naturally in this context is the idea of a Gleason part. Various characterizations of parts are obtained as a preliminary to the embedding theorem. Browder's proof of Wermer's theorem is along the lines of the original proof. Gamelin gives two proofs of this theorem and gives further generalizations of the embedding theorem. His chapter on parts also contains a detailed analysis of the parts of the algebra $R(X)$ and a discussion of the Farrell-Rubel-Shields theorem on pointwise bounded approximation.

Given a uniform algebra A and a measure M that is multiplicative on A, one can take the closure of A in $L^p(m)$ (weak* closure if $p=\infty$) to obtain the Hardy spaces H^p ($1 \leqq p \leqq \infty$). The usefulness of these spaces goes beyond the fact they provide interesting examples of Banach spaces. They can be used to obtain information on the algebra A itself. Wermer's proof of his embedding theorem is an example of this phenomenon. Another reason for studying these spaces is their close relation to a problem of weighted trigonometric approximation. This is the prediction problem for stationary processes in Hilbert space. In fact this application may be used very effectively to motivate the theory of Hardy spaces. Of the books under review, only Lumer's notes contain details of this application to probability theory.

There are two basic approaches to the study of Hardy spaces. Both of them were introduced in the classic papers of Helson and Lowdenslager. The first approach relies heavily on the solution of an extremal problem of Szegö. Although Helson and Lowdenslager used this method in the context of Hardy spaces on certain compact groups, the method can be refined to take care of Hardy spaces of more general algebras. This has been done in the works of Ahern and Sarason, Hoffman, König, Lumer, and Srinivasan and Wang. In his book, Browder restricts his attention to Hardy spaces of uniform algebras with unique representing measures and follows the approach based on Szegö's theorem. The second approach to Hardy space theory is based on abstract conjugate function theory. Here the basic tools are Bochner's generalizations of the conjugate function theorems of M. Riesz and Kolmogorov. This approach reached a definitive stage in the paper of Gamelin and Lumer. Both Gamelin and Lumer base their development of Hardy space theory on the universal Hardy class and the abstract conjugation operator. They do not insist on unique representing measures. They are able to obtain some very interesting results when the set of representing measures is finitely generated. This generalization becomes necessary to study analytic function algebras on finitely connected sets. The main results obtained in Hardy space theory generalize the results of Beurling on Hardy spaces of the circle. These include the structure theory of invariant subspaces and the inner-outer factoring of elements of H^p.

When the algebra under consideration is that associated with a compact abelian group with ordered dual, more precise information on the Hardy spaces can be obtained. Gamelin has an excellent chapter devoted to this topic. Most of the material of this chapter is drawn from the works of Arens, Helson, Hoffman, Lowdenslager and

Singer. The Hoffman-Singer theory of maximal algebras and the Helson-Lowdenslager work on invariant subspaces and cocylces is made available for the first time in book form. Also included are extensions of the works of these authors due to deLeeuw and Glicksberg and Gamelin.

This review would be incomplete without a few words on the merits of these books as textbooks. Browder's book seems ideal for a one-semester course for students who already know some function theory and basic Banach space theory. Because of the more detailed treatment, the student may find it easier to read Browder. The first two chapters of Gamelin can be also used as material for an introductory course. The later chapters offer a magnificent selection of topics that can be offered in specialized courses. Browder's book does not offer any exercises. It also lacks a terminological index. Gamelin's book contains exercises of varying degrees of complexity. Some of the problems are actually theorems from recent papers. In such cases the author purposely omits the references. The reviewer feels that it would have been nicer to give references to some of the more difficult exercises.

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Mordell, Diophantine equations, by L. J. Mordell, Academic Press, New York and London, 1969.

The theory of diophantine equations is one of the oldest in mathematics, one of its most attractive, and also at the moment one which is still fairly undeveloped as being exceptionally hard. One reason for this is perhaps that in the full generality of the Hilbert problem, it cannot be effectively dealt with. Nevertheless, I personally would expect a wide class of diophantine problems to be effectively solvable (e.g. those on curves or abelian varieties), and in any case, many special cases are solvable.

Because of difficulties which have been encountered historically, a portion of the subject has developed as an accumulation of special diophantine equations, mostly in two variables, i.e. curves. It was well understood in the nineteenth century that nonsingular cubic curves have a group law on them, parametrized by the elliptic functions from a complex torus, but Poincaré was the first to draw attention to the special group of rational points when this curve is defined by an equation with rational coefficients, and he guessed that this group might be finitely generated. Mordell proved this fact in 1922, and thereby provided the first opportunity to behold the beginnings of a much broader approach to this type of equation. He