THE EXACT SEQUENCE OF LOW DEGREE AND NORMAL ALGEBRAS

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The exact sequence of low degree associated to a first quadrant bicomplex (five terms long in [4, I.4.5.1] seven terms long in [2, Lemma 7.5]) has been used in a number of situations, for example, in obtaining a cohomological description of the Brauer group of a commutative ring R [2]. In this note we observe that the sequence may be extended to an infinitely long exact sequence. The terms arising from the homology of the total complex are not $F^{n-1}H^n(\text{tot})$, the $(n-1)$th filtration group of H^n, for $n>2$, but map onto it.

As an application we embed the seven term Galois cohomology sequence of [1, 5.5] into an infinite sequence, and sketch a map from normal Azumaya algebras into the eighth term which extends the Teichmüller cocycle map of [3].

1. Suppose given a bicomplex $\{C_{p,q}\}$ of abelian groups [5, p. 340] such that $C_{p,q}=0$ if $p<0$ or $q<0$. The differentials $d': C_{p,q}\to C_{p+1,q}$ and $d'': C_{p,q}\to C_{p,q+1}$ of the bicomplex, defined for all integers p, q, satisfy the conditions $d'd'=0$, $d''d''=0$, $d'd''+d''d'=0$. (Notation: cl( ) will denote "cohomology class of.") Then $Z^2_{p,q}$ is the set of classes $\text{cl}(u)$ in $\ker(d'')/\text{im}(d'')$ such that u is in $C_{p,q}$, $d''(u)=0$, and $d'(u)=d''(v)$ for some v in $C_{p+1,q-1}$; $B^2_{p,q}$ is the set of classes $\text{cl}(u)$ such that u in $C_{p,q}$ is of the form $u=d'(v)+d''(w)$ with $d''(v)=0$; and $E^2_{p,q}=Z^2_{p,q}/B^2_{p,q}$.

The nth group $C_n(\text{tot})$ of the total complex $(n\geqq 0)$ is the group $C_{0,n}\oplus C_{1,n-1}\oplus\cdots\oplus C_{n-1,1}\oplus C_{n,0}$. Set $D=d''+d'$, the differential of the total complex $\{C_n(\text{tot})\}$. Denote by Z^n the elements of $C_n(\text{tot})$ of the form $x=(0,\cdots,0,u,v)$ with $Dx=0$. Denote by $\dot{B}^n$ the elements of Z^n of the form $x=Dy$, where $y=(0,\cdots,0,z_1,z_0)\in C_{n-1}(\text{tot})$, and B^n the elements of Z^n of the form $x=Dy$ where $y=(z_n,z_{n-1},\cdots,z_2,z_1,z_0)\in C_{n-1}(\text{tot})$. Then the filtered group $F^{n-1}H^n$ of the total complex associated to the bicomplex is Z^n/B^n. We denote by $\dot{H}^n$ the group $Z^n/\dot{B}^n$. Note that there is clearly an epimorphism from $\dot{H}^n$ onto $F^{n-1}H^n$ for all n, and for $n=2$ or 1 it is the

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identity map.

THEOREM 1. *The long exact sequence of low degree is*

$$0 \to E^2_{1,0} \to \dot{H}^1 \to E^2_{0,1} \to E^2_{2,0} \to \dot{H}^2 \to E^2_{1,1} \to \cdots$$
$$\cdots \to E^2_{n,0} \to \dot{H}^n \to E^2_{n-1,1} \to E^2_{n+1,0} \to \dot{H}^{n+1} \to \cdots.$$

The first seven terms of this sequence are identical with the sequences of [2, Lemma 7.5].

The maps are as follows:

From $E^2_{n,0}$ to $\dot{H}^n$ the map is obtained by sending cl(cl(v)) in $E^2_{n,0}$ to the class of the element $(0, \cdots, 0, v)$ of Z^n. From $\dot{H}^n$ to $E^2_{n-1,1}$ the map is obtained by sending the class of an element $(0, \cdots, 0, u, v)$ of Z^n to cl(cl(u)) in $E^2_{n-1,1}$. From $E^2_{n-1,1}$ to $E^2_{n+1,0}$ the map is obtained by sending cl(cl(u)) in the former to cl(cl(w)) in the latter, where $d'(u) = d''(z)$ for some z in $C_{n,0}$ and $w = d'(z)$.

The proof of exactness is a routine computation.

Alternatively, one can obtain the sequence by forming the bicomplex $\{\dot{C}_{p,q}\}: \dot{C}_{p,q} = C_{p,q}$ for $q \leqq 1$, $\dot{C}_{p,q} = 0$ for $q \geqq 3$, and $\dot{C}_{p,2} = d''(C_{p,1})$. Then one has a long exact sequence by [6, Corollary 2.3], whose E^2 terms are the same as those in Theorem 1 and it is not difficult to show that the H^n(tot) of the bicomplex $\dot{C}_{p,q}$ occurring in this latter sequence is the same as the $\dot{H}^n$ of Theorem 1. We omit details in either case.

2. We now sketch a generalization of the Teichmuller cocycle map for normal central simple algebras described by Eilenberg and MacLane [3]. For Amitsur cohomology and other unexplained notation see [2].

Let S be a commutative ring with unit, and a Galois extension of R with group Q [1]; let A be a Azumaya (=central separable) S-algebra. Suppose A is Q-normal, i.e. there is a 1-1 set map w from Q to $\mathrm{Aut}_R(A)$ whose image restricts to Q on S.

Let K be a commutative R-algebra which is a faithfully flat R-module. Let U_S be the functor from R-algebras to abelian groups defined by $U_S(K)$ = the multiplicative group of units of $S \otimes_R K$. Let $C_{p,q} = (Q^p, U_S(K^{q+1}))$, with $p, q \geqq 0$, where (——, ——) denotes set maps, Q^p denotes direct product p times, $Q^0 = \{1\}$, K^{q+1} denotes tensor product over R $q+1$ times. Let d_Q denote the group cohomology coboundary (in the p-direction), and d'_K the Amitsur cohomology coboundary (in the q-direction). Then $d'_K d_Q = d_Q d'_K$, so replacing d'_K by $s^p \circ d' = d_K$ on $C_{p,q}$, where s^p denotes the inverse map in U_S applied p times, we obtain a bicomplex. From it we obtain a long exact sequence of low degree. Part of it reads

$$\cdots \to H^1(Q,\, H^1(K/R,\, U_S)) \to H^3(Q,\, H^0(K/R,\, U_S))$$
$$\to \dot{H}^3(\mathrm{tot}) \to H^2(Q,\, H^1(K/R,\, U_S)) \to \cdots.$$

(We note that $H^1(K/R,\, U_S)$, Amitsur cohomology, is isomorphic to the kernel of the map from $\mathrm{Pic}(S)$ to $\mathrm{Pic}(S\otimes_R K)$ by descent, and $H^0(K/R,\, U_S) = U_S(R) = U(S)$, so that taking limits over faithfully flat K (as in [**2**, §6]) yields, except for identifying inj lim $\dot{H}^2(\mathrm{tot})$ with $B(S/R)$, an extension of the Chase-Harrison-Rosenberg-Auslander-Brumer seven term exact sequence of Galois cohomology [**1**, 5.5]. Note that inj lim $\dot{H}^3(\mathrm{tot})$ is the first term missing from [**1**, 5.5].)

Cocycles in Z^3 corresponding to elements of $\dot{H}^3(\mathrm{tot})$ are pairs (f, g), f in $(Q^2,\, U_S(K^2))$, g in $(Q^3,\, U_S(K))$ where $d_K(f) = 1$, $d'_K(g) = d_Q(f)$ and $d_Q(g) = 1$.

THEOREM 2. *There is a well-defined map from the set of isomorphism classes of Q-normal Azumaya S-algebras into* inj $\lim_K \dot{H}^3(\mathrm{tot})$. *When S is semilocal $\dot{H}^3(\mathrm{tot}) \cong \dot{H}^3(G,\, U(S))$ and the composition is the Teichmuller cocycle map of* [**3**].

We shall restrict ourselves here to sketching the definition of the map, and leave checking of the well-definedness (which is not difficult) and elaboration of the theory for future appearance elsewhere.

The map is defined as follows. Let $w: Q \to \mathrm{Aut}_R(A)$ be an extension of Q to A. Then $dw(\lambda, \mu) = w(\lambda)w(\mu)w(\lambda, \mu)^{-1}$ fixes S so is an element of $\mathrm{Aut}_S(A)$. We have a map $J: \mathrm{Aut}_S(A) \to \mathrm{Pic}(S)$ defined by: f goes to $J_f = \{a \text{ in } A \mid f(x)a = ax \text{ for all } x \text{ in } A\}$. So dw yields a map J_{dw} from Q^2 to $\mathrm{Pic}(S)$. (See [**7**].)

Let K be a faithfully flat R-algebra such that $S\otimes_R K$ splits all the images of J_{dw}. Viewing dw as mapping Q^2 into $\mathrm{Aut}_{S\otimes K}(A\otimes K)$, since $J_{dw(\lambda,\mu)} \otimes K$ is a free $S\otimes K$ module with basis element $u(\lambda, \mu)$ in $A\otimes K$, $dw(\lambda, \mu)$ is conjugation by $u(\lambda, \mu)$ in $A\otimes K$ [**7**].

An element f of $(Q^2,\, U(S\otimes K^2))$ corresponding to A is the Amitsur cocycle corresponding to the modules $J_{dw(\lambda,\mu)}$, namely $d'_K(u)$ where

$$d'_K(u(\lambda, \mu)) = \epsilon_1(u(\lambda, \mu))^{-1}\epsilon_0(u(\lambda, \mu)).$$

On the other hand, an element g of $(Q^3,\, U_S(K))$ corresponding to A is the noncommutative coboundary $d_Q(u)$, defined exactly as the Teichmüller cocycle is defined in the classical case [**3**], viz:

$$d_Q(u)(\lambda, \mu, \nu) = u(\mu, \nu)^{w(\lambda)}u(\lambda, \mu\nu)u^{-1}(\lambda\mu, \nu)u^{-1}(\lambda, \mu).$$

It is not difficult to show that $d'_K(d_Q(u)) = d_Q(d'_K(u))$, so that $(d_K(u), d_Q(u))$ is an element of Z^3, whose class in $\dot{H}^3$ is the image of A under the map of Theorem 2.

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