THE EXISTENCE OF FREE METACYCLIC ACTIONS ON HOMOTOPY SPHERES

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This question is rather old [3]: Which groups having periodic cohomology can act freely on some homotopy sphere?

The first nontrivial restriction was supplied by Milnor in [3]. Every element of order two must lie in the center. Swan [6] showed that any group with periodic cohomology acts freely on a c.w. complex of the homotopy type of sphere.

If the group has odd order then it is metacyclic. The main result of this note is that most metacyclic groups of odd order *can* act freely and smoothly on some homotopy sphere (Corollary 6). Of independent interest is the discussion of the algebraic tools used in the solution.

I. **The algebraic tools.** Let π be a finite group and C_k be the category of pairs (M, φ) where

(1) M is a left $Z[\pi]$ module of homological dimension $\leqq 1$ and M as an abelian group has finite order and no two torsion.

(2) $\varphi: M \to \mathrm{Hom}_Z(M, Q/Z)$ is a $(-1)^k$ Hermitian form over $Z[\pi]$, i.e. φ is an isomorphism of $Z[\pi]$ modules and $\varphi(x, y) = (-1)^k \varphi(y, x)$, $x, y \in M$. $\mathrm{Hom}_Z(M, Q/Z)$ is made into an abelian group by $(\lambda f)(x) = f(\bar{\lambda} x)$ for $f \in \mathrm{Hom}_Z(M, Q/Z)$, $x \in M$, $\lambda \in Z[\pi]$ and $\bar{\lambda}$ is the conjugate of λ under the involution on $Z[\pi]$ defined by sending group elements into their inverses.

(3) $0 \to (M_1, \varphi_1) \to (M_2, \varphi_2) \to (M_3, \varphi_3) \to 0$ is exact in C_k iff $M = M_1 \oplus M_2$, $\varphi = \varphi_1 \oplus \varphi_2$.

Certain elements of C_k are to be regarded as trivial. These are obtained in this fashion: Let F be a free $Z[\pi]$ module of rank n and A an $n \times n$ matrix over $Z[\pi]$ satisfying $\bar{A}_{ij} = (-1)^k A_{ji}$. Define M by the exact sequence $0 \to F \xrightarrow{A} F \xrightarrow{\omega} M \to 0$ so that M is the cokernel of A. Suppose M is finite. Define a $(-1)^k$ Hermitian form φ on M by

$$\phi(\omega(x), \omega(y)) = \sum_{i,j} x_i \cdot \overline{A_{ij}^{-t}} \cdot \bar{y}_j \mod Z[\pi].$$

Explanation! Since M is finite, A has an inverse A^{-1} over $Q[\pi]$. Its transpose is A^{-t}. If $e_1 \cdots e_n$ is a base for F then $x = \sum x_i e_i$ and

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$y = \sum y_j e_j$. Now the above expression makes sense once one observes that

$$\mathrm{Hom}_Z (M, Q/Z) \cong \mathrm{Hom}_{Z(\pi)} (M, Q(\pi)/Z(\pi)).$$

There are two geometrically significant groups which can be fashioned from this data. The first denoted by $\mathcal{L}_k(\pi)$ is the Grothendieck group of C_k modulo the subgroup generated by the trivial elements of C_k. The second denoted by $G_k(\pi)$ is the Grothendieck group of a subcategory C_k' modulo the subgroup of trivial elements. The category C_k' consists of the subset of C_k of elements (M, φ) such that $(M \oplus M, \varphi \oplus -\varphi)$ is trivial. Here $[-\varphi](x, y) = -\varphi(x, y)$.

II. **Geometric motivation.** The geometric relevance of these groups is this: If M and N are closed oriented manifolds of dimension $2k-1$, $f: M \to N$ is a normal map, [2], $\pi_1(N)$ is finite of odd order and the kernel of f_*, $K_*(f)$, is zero in dimensions $\neq k-1$ and if $K_{k-1}(f)$ is a finite group, then the linking number form [2], [7] on $H_{k-1}(M)$ provides a $(-1)^k$ Hermitian form $\varphi: K_{k-1}(f) \to \mathrm{Hom}_Z(K_{k-1}(f), Q/Z)$. Surgery is possible on M and $K_{k-1}(f)$ can be killed iff $(K_{k-1}(f), \varphi) \in G_k(\pi)$ is zero [7]. The geometry of the situation guarantees that $(K_{k-1}(f), \varphi) + (K_{k-1}(f), -\varphi)$ is trivial [7].

The group $\mathcal{L}_k(\pi)$ is more closely related to the aim of this paper, especially when π has periodic cohomology. Specifically, suppose that π has periodic cohomology of period $2k$ and that V^{2k-1} is a stably parallelizable manifold of dimension $2k-1$ supporting a free action of π. Then V^{2k-1} determines an element $[V]$ in $\Omega_{2k-1}^{SO}(B_\pi)$, the oriented bordism group of the classifying space of π.

Question. Is $[V]$ equal to $[\Sigma]$ where Σ is a $(2k-1)$-dimensional homotopy sphere supporting a free action of π?

A necessary condition is that

$$\text{(P)} \qquad \mu[V] \in H_{2k-1}(B_\pi) \text{ has order } n = \text{order } \pi.$$

Here μ is the natural transformation from bordism to homology. We can assume that V is $k-2$ connected. If $H_{k-1}(V)$ is finite then (P) implies that the homological dimension of $H_{k-1}(V)$ is ≤ 1. Then the linking number form φ on $H_{k-1}(V)$ defines an element $(H_{k-1}(V), \varphi)$ in $\mathcal{L}_k(\pi)$ whose vanishing is a necessary and sufficient condition (in the case π is of odd order) that $[V] = [\Sigma]$ in $\Omega_{2k-1}^{SO}(B_\pi)$ for some Σ.

Let me remark that there is a natural homomorphism Δ from $\mathcal{L}_{2k}(\pi)$ to $\tilde{K}_0(Z(\pi))$, the reduced projective class group of $Z(\pi)$ obtained by sending (M, φ) to P where $0 \to P \to F \to M \to 0$ is an exact sequence over $Z(\pi)$, F is free and P is projective. Let $\mathcal{L}_k^0(\pi) = \text{kernel } \Delta$.

III. **Algebraic results.** Having motivated the usefulness of these groups, let me record my present knowledge about them.

The first theorem is quite general and surprisingly simple to prove.

THEOREM 1. *For any finite* π, $G_k(\pi)$ *has exponent* 4.

In order to state the next result let us introduce the notation $Z_{p,q}$ for the metacyclic group

$$\{x, y \mid x^p = y^q = 1, yxy^{-1} = x^\sigma \sigma^q \equiv 1 \ (p)\}$$

where p is odd and q is an odd prime.

Let (Σ) denote the ideal of $Z_{q^n}(P)$ generated by $\sum_{i=0}^{p-1} x^i$. Here $P \subset Z_{p,q}$ is the subgroup generated by x. Then $\Gamma = Z_{q^n}(P)/(\Sigma)$ is a module over $Z(Z_{p,q})$. The action of $Z_{p,q}$ on Γ is defined by the fact that Γ is a $Z(\pi_p)$ module and defining $y(\sum a_i x^i) = \sum a_i x^{\sigma i}$.

Also if Q denotes the cyclic group of order q generated by y, then $Z_{q^n}(Q)$ is a module over $Z[\pi]$ in an obvious way.

THEOREM 2. *Up to equivalence there is a unique* -1 *Hermitian form on each of* Γ *and* $Z_{q^n}(Q)$.

Let these forms be denoted by φ_n and ψ_n respectively. Set $x_n = \Gamma, \varphi_n)$ and $y_n = (Z_{q^n}(Q), \psi_n)$.

PROPOSITION 3. $\Delta x_n = n\Delta x_1$, $\Delta y_n = n\Delta y_1$.

REMARK. $\tilde{K}_0(Z(\pi))$ is a finite group for any finite π (Swan [5]). This means that for some n, $x_n, y_n \in \mathcal{L}_1^0(\pi)$.

IV. **A serious application.** We now arrive at the first serious application of these ideas. Let $Z_{p,q}$ be as above and P the normal subgroup generated by x. Then there is an exact sequence

$$1 \to P \xrightarrow{i} Z_{p,q} \xrightarrow{j} Q \to 1.$$

Let V_P and V_Q be one-dimensional complex representations of P and Q respectively, defined by multiplication by a primitive pth root of unity ξ_p, respectively a primitive qth root of unity ξ_q. Then

$V = i_* V_p \oplus j^* V_q$ is a complex $(q+1)$-dimensional representation of $Z_{p,q}$. Here $i_* V_p$ is the induced representation of $Z_{p,q}$ and $j^* V_q$ is the one-dimensional representation of $Z_{p,q}$ defined by j.

If $z_1, z_2, \cdots, z_q$ are complex coordinates for $i_* V_p$ and z_{q+1} is a complex coordinate for $j^* V_q$, then the polynomial $f_n(z_1, z_2, \cdots, z_{q+1}) = \sum_{i=1}^q z_i^p + z_{q+1}^{q^n}$ on V is invariant under the action of π. Set $K_{f_n} = \{z \mid f_n(z) = \epsilon, \|z\| = \eta\}$ where ϵ and η are positive numbers so chosen that π acts freely on K_{f_n}.

Then K_{f_n} is $q-2$ connected [4] and $H_{q-1}(K_{f_n})=N\Gamma \oplus MZ_{q^n}(Q)$ as a $Z(Z_{p,q})$ module. Fortunately N and M are even integers.

Set $M_n=H_{q-1}(K_{f_n})$ and ω_n the linking number form on M_n. Then since q is odd $(M_n, \omega_n)\in\mathcal{L}_1(z_{p,q})$. By Theorem 2, $(M_n, \omega_n)=Nx_n+My_n$.

THEOREM 4. *If x_n, $y_n\in\mathcal{L}_1^0(\pi)$, then their exponent is* 2. *For an appropriate n, the hypothesis is satisfied.*

Since N and M are even we have

COROLLARY 5. *If $Z_{p,q}$ is the metacyclic group where p is odd and q is an odd prime, then $Z_{p,q}$ acts freely and smoothly on a homotopy sphere of dimension* $2q-1$.

REMARK. The period of $Z_{p,q}$ is $2q$ in this case so that $2q-1$ is the minimal dimension of a homotopy sphere supporting a free $Z_{p,q}$ action.

REMARK. R. Lee has obtained a remarkable algebraic result which implies a weakened form of Corollary 6. His result applies to the group $Z_{p,q}$ where p and q are both prime.

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