SOME INTRICATE NONINVERTIBLE LINKS

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Let L be an oriented, ordered link imbedded in the oriented 3-sphere S^3, and let μ and κ be integers such that $1 \leqq \kappa < \mu$. We say that L is a *generalized noninvertible link for the pair* μ, κ (or a $(\mu, \kappa)I$ *link*) if it satisfies:

(i) *L has μ components*;

(ii) *Each sublink with κ or fewer components is invertible*;

(iii) *Each sublink with more than κ components is noninvertible.*

L is *invertible* provided it is of the same (oriented) type as its inverse. The *inverse* of L is obtained by reversing the orientation of each component of L.

Now $(2, 1)I$ links were exhibited in [2] and a $(\mu, \mu-1)I$ link was given in [3] for each $\mu \geqq 3$. In this announcement we outline the construction of a generalized noninvertible link for each pair μ, κ such that $1 \leqq \kappa < \mu$ and $\mu \geqq 3$. Details will appear elsewhere.

1. **Two propositions.** The following propositions clear the way for the constructive type proof of the main Theorem 2.1. An induction argument together with results of [2] yields a proof of

PROPOSITION 1.1. *For each integer $\mu \geqq 2$, there exists a $(\mu, 1)I$ link in S^3.*

The combined contents of [2] and [3] are stated in

PROPOSITION 1.2. *For each integer $\mu \geqq 2$, there exists a $(\mu, \mu-1)I$ link in S^3.*

2. **$(\mu, \kappa)I$ links.** The main result is

THEOREM 2.1. *For each pair of integers μ, κ such that $1 \leqq \kappa < \mu$, there is a generalized noninvertible link $\mathfrak{L}$ in S^3 satisfying* (i), (ii), *and* (iii) *of the introduction.*

OUTLINE OF CONSTRUCTION. By Propositions 1.1 and 1.2, we need consider only those integers μ, κ for which $2 \leqq \kappa < \mu - 1$. We relax this, however, and assume only that $2 \leqq \kappa < \mu$.

Set $\nu = \binom{\mu-1}{\kappa}$. Let $Q_1, \ldots, Q_\nu$ be a collection of disjoint 3-cells in S^3

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each of which is in the shape of a solid cylinder. In each Q_l, ($l=1, \ldots, \nu$), construct the oriented, ordered link

$$L_l = (l, 1) \cup \cdots \cup (l, \alpha_{l2} - 1) \cup (l, \alpha_{l2}) \cup \cdots \cup (l, \alpha_{l\kappa+1})$$

as shown in Figure 1. (Two small arcs of each component are to lie on ∂Q_l as indicated with the remainder of L_l in Int Q_l.) The set $\{\alpha_{l2}, \ldots, \alpha_{l\kappa+1}\}$ is the lth combination of the integers $2, \ldots, \mu$ taken κ at a time, and in the lexicographical ordering of these combinations.

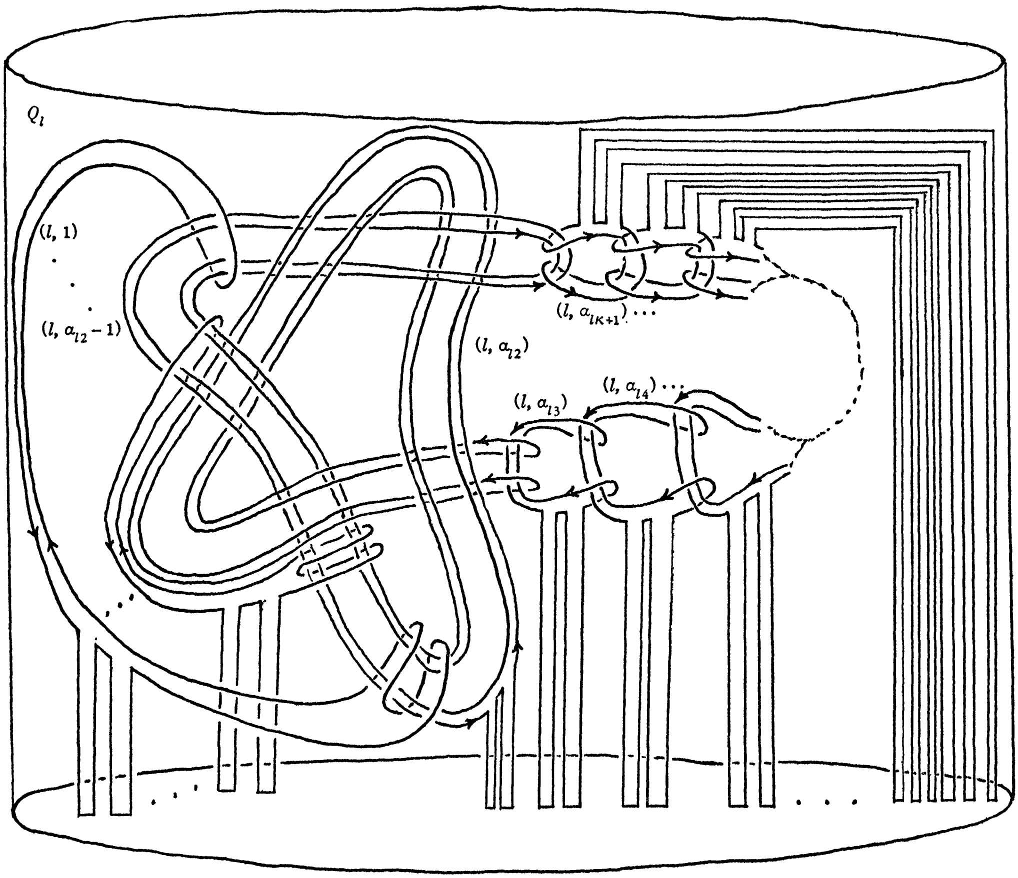

FIGURE 1

Now let $(l_1, \alpha), \ldots, (l_{t(\alpha)}, \alpha)$ be the collection of all those pairs whose second coordinate is α. We assume that $l_1 < \ldots < l_{t(\alpha)}$, set $\mathcal{K}_\alpha = (l_1, \alpha) \# \ldots \# (l_{t(\alpha)}, \alpha)$, and $\mathcal{L} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_\mu$. (R. H. Fox gives a nice account of the composition operation # in §7 of [1].) The compositions, formed inductively with respect to α, are to be made by running two parallel arcs (each with proper orientation) in the obvious nice way from (l_m, α) to (l_{m+1}, α), $(m=1, \ldots, t(\alpha)-1)$, and then deleting the appropriate small arcs on ∂Q_{l_m} and $\partial Q_{l_{m+1}}$. Several routine requirements on the placements of these pairs of arcs are also made.

That $\mathfrak{L}$ is a $(\mu, \kappa)I$ link follows from the construction and the following properties of L_l in Figure 1:

1. For each $j=1, \ldots, \alpha_{l2}-1$, the sublink $(l, j)\cup(l, \alpha_{l2})\cup \ldots \cup (l, \alpha_{l\kappa+1})$ is a $(\kappa+1, \kappa)I$ link. Methods similar to those of [3] prove this.

2. Any sublink of L_l which is obtained by removal of one of the components $(l, \alpha_{l2}), \ldots, (l, \alpha_{lK+1})$ is completely splittable.

References

1. R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961) Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 120–167. MR 25 #3522.

2. W. Whitten, *A pair of non-invertible links*, Duke Math. J. **36** (1969), 695–698.

3. ———, *On noninvertible links with invertible proper sublinks*, Proc. Amer. Math. Soc. (to appear).

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