EXTENSION THEORY FOR CONNECTED HOPF ALGEBRAS

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1. **Introduction.** Let K be a fixed commutative ring with unit. We will deal with graded algebras, coalgebras, and Hopf algebras over K as defined in Milnor-Moore [4], but we assume that the underlying K-modules are connected.

Suppose A, B are Hopf algebras; A commutative and B cocommutative. By an extension of B by A we mean a diagram of Hopf algebras and Hopf maps

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B \tag{1.1}$$

in which C is isomorphic to $A \otimes B$ simultaneously as a left A-module and right B-comodule. In this paper we announce results which describe and classify all extensions by B by A. Proofs will appear in [5].

2. **Matched pairs.** If B is an algebra we write

$$\eta: K \to B, \qquad \mu_B: B \otimes B \to B$$

for the unit and multiplication, respectively. If A is a coalgebra we write

$$\epsilon: A \to K, \qquad \psi_A: A \to A \otimes A$$

for the counit and comultiplication.

As the first step in classifying extensions, we will show in [5] how a diagram (1.1) determines a pair of K-linear maps

$$\sigma_A: B \otimes A \to A, \qquad \rho_B: B \to A.$$

σ_A is the "action" of base on fiber that one expects in an extension problem; ρ_B is its dual. We prove:

(a) σ_A gives A the structure of a left B-module algebra;

(b) ρ_B gives B the structure of a right A-comodule coalgebra;

(c) the diagram commutes:

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$$
(2.1)\qquad
\begin{tikzcd}
B \otimes A \arrow[r, "\sigma_A"] \arrow[d, "\psi_B \otimes \psi_A"] & A \arrow[r, "\psi_A"] & A \otimes A \\
B \otimes B \otimes A \otimes A \arrow[d, "\rho_B \otimes B \otimes \mathrm{A} \otimes \mathrm{A}"] & & \\
B \otimes A \otimes B \otimes A \otimes A \arrow[d, "{(1, 4, 2, 3, 5)}"] & & \\
B \otimes A \otimes A \otimes B \otimes A \arrow[rr, "\sigma_A \otimes \mathrm{A} \otimes \sigma_A"] & & A \otimes A \otimes A \arrow[uuu, "A \otimes \mu_A"']
\end{tikzcd}
$$

(d) the diagram commutes:

$$
(2.2)\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\mu_B"] \arrow[d, "\psi_B \otimes B"] & B \arrow[r, "\rho_B"] & B \otimes A \\
B \otimes B \otimes B \arrow[d, "\rho_B \otimes \mathrm{B} \otimes \rho_B"] & & \\
B \otimes A \otimes B \otimes B \otimes A \arrow[d, "{(1, 4, 2, 3, 5)}"] & & \\
B \otimes B \otimes A \otimes B \otimes A \arrow[rr, "B \otimes B \otimes A \otimes \sigma_A"] & & B \otimes B \otimes A \otimes A \arrow[uuu, "\mu_B \otimes \mu_A"']
\end{tikzcd}
$$

Conversely, suppose given a pair of Hopf algebras (A, B) with A commutative and B cocommutative, and suppose given K-linear maps σ_A, ρ_B satisfying conditions (a)–(d). Then we call (A, B) a "matched pair".

3. Bimodules over a matched pair. Suppose (A, B) is a matched pair. Suppose N is simultaneously a left B-module under $\sigma_N: B \otimes N \to N$, and a right A-comodule under

$$\rho_N: N \to N \otimes A.$$

Then we call N an (A, B)-bimodule if the diagram commutes:

$$
\begin{tikzcd}
B \otimes N \arrow[r, "\sigma_N"] \arrow[d, "\psi_B \otimes N"] & N \arrow[r, "\rho_N"] & N \otimes A \\
B \otimes B \otimes N \arrow[d, "\rho_B \otimes B \otimes \rho_N"] & & \\
B \otimes A \otimes B \otimes N \otimes A \arrow[d, "{(1, 4, 2, 3, 5)}"] & & \\
B \otimes N \otimes A \otimes B \otimes A \arrow[rr, "\sigma_N \otimes A \otimes \sigma_A"] & & N \otimes A \otimes A \arrow[uuu, "N \otimes \mu_A"']
\end{tikzcd}
\tag{3.1}
$$

For example, if (A, B) is a matched pair we can give A the structure of an (A, B)-bimodule, with left B-action $\sigma_A: B \otimes A \rightarrow A$, and right A-coaction $\psi_A: A \rightarrow A \otimes A$. A dual construction makes B into an (A, B)-bimodule. We say $f: M \rightarrow N$ is a map of (A, B)-bimodules if f is simultaneously a map of left B-modules and right A-comodules.

The interpretation of diagrams (2.1), (2.2), (3.1) is found in:

THEOREM 3.1. *Let N be a bimodule over the matched pair (A, B). Let the map $\tilde{\sigma}_{N \otimes A}: B \otimes N \otimes A \rightarrow N \otimes A$ be the composition*

$$(N \otimes \mu_A)(\sigma_N \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5)(\rho_B \otimes B \otimes N \otimes A)(\psi_B \otimes N \otimes A).$$

Let the map $\tilde{\rho}_{N \otimes A}: N \otimes A \rightarrow N \otimes A \otimes A$ be $N \otimes \psi_A$. Then:

(a) *$\tilde{\sigma}_{N \otimes A}$ gives $N \otimes A$ the structure of a left B-module;*

(b) *$\tilde{\rho}_{N \otimes A}$ gives $N \otimes A$ the structure of a right A-comodule;*

(c) *with these structure maps $N \otimes A$ is in fact an (A, B) bimodule which we denote $N \tilde{\otimes} A$;*

(d) *$\rho_N: N \rightarrow N \tilde{\otimes} A$ is a map of (A, B)-bimodules.*

Theorem 3.1 has a dual. Given an (A, B)-bimodule M, Theorem (3.1)* tells how to give $B \otimes M$ the structure of an (A, B)-bimodule, denoted $B \underset{\sim}{\otimes} M$, in such a way that $\sigma_M: B \underset{\sim}{\otimes} M \rightarrow M$ is a map of (A, B)-bimodules.

4. (A, B)**-algebras and** (A, B)**-coalgebras.** By an (A, B)-algebra we mean an (A, B)-bimodule N that is also a commutative algebra, in such a way that $\mu_N: N \otimes N \rightarrow N$ is both a map of left B-modules and right A-comodules. (A, B)-coalgebras are defined similarly. For example if (A, B) is a matched pair, then A itself is an (A, B)-algebra, and B is an (A, B)-coalgebra.

THEOREM 4.1. *Let N be an (A, B)-algebra. Let $N \tilde{\otimes} A$ have the algebra structure of the tensor product $N \otimes A$. Then $N \tilde{\otimes} A$ is an (A, B)-algebra, and $\rho_N: N \rightarrow N \tilde{\otimes} A$ is a map of (A, B)-algebras.*

Theorem 4.1 can be interpreted in the language of "triples" [1], [2]. Let Γ be the category of (A, B)-algebras. Let $S\colon\Gamma\to\Gamma$ be the functor which carries N to $N\tilde{\otimes}A$, and let $I\colon\Gamma\to\Gamma$ be the identity functor. Define functor transforms $\delta\colon I\to S$, $\sigma\colon S^2\to S$ by

$$\delta(N)=\rho_N\colon N\to N\tilde{\otimes}A;\quad \sigma(N)=N\otimes\epsilon\otimes A\colon N\tilde{\otimes}A\tilde{\otimes}A\to N\tilde{\otimes}A.$$

Then $V\equiv(S,\delta,\sigma)$ is a triple on the category Γ.

Similarly, the dual of Theorem 4.1 gives a cotriple $W\equiv(T, d, s)$ on the category Δ of (A, B)-coalgebras. Here $T(M)=B\underset{\sim}{\otimes}M$, $d(M)=\sigma_M\colon B\underset{\sim}{\otimes}M\to M$, and $S(M)=B\otimes\eta\otimes M\colon B\underset{\sim}{\otimes}M\to B\underset{\sim}{\otimes}B\underset{\sim}{\otimes}M$.

5. **The cohomology of matched pairs.** If ϕ is any category, write $S^*(\phi)$ for the category of cosimplicial objects over ϕ, and $S_*(\phi)$ for the category of simplicial objects over ϕ. Then the triple V of §4 gives rise in the usual way [1], [2] to a functor $V\colon\Gamma\to S^*\Gamma$; the cotriple W gives rise to a functor $W\colon\Delta\to S_*\Delta$. For example, if (A, B) is a matched pair, then $V(K)$ is the acyclic cobar construction on A, but it has some structure not present in the classical case . . . an action of B compatible with the coface operators. Similarly, $W(K)$ is the acyclic bar construction on B, with an A-coaction added.

If M is an (A, B)-coalgebra and N an (A, B)-algebra, let $\mathrm{Hom}_{(A,B)}(M, N)$ denote the set of maps of (A, B)-modules $f\colon M\to N$ for which $f_0\colon M_0\to N_0$ is the identity on K. Then $\mathrm{Hom}_{(A,B)}(M, N)$ is an abelian group under the composition law $f+g=\mu_N(f\otimes g)\psi_M$.

Now to any matched pair (A, B) we associate a double cosimplicial abelian group $X(B, A)$ by setting:

$$X^{p,q}(B, A) = \underset{(A,B)}{\mathrm{Hom}}\,(W(K)_p, V(K)^q). \tag{5.1}$$

Let $\overline{X}(B, A)$ be the associated "total" cochain complex. Then we define the cohomology of the matched pair (A, B) by:

$$\overline{H}^*(B, A) = H^*(\overline{X}(B, A)). \tag{5.2}$$

6. **Classification of extensions.** Let (A, B) be a matched pair under (σ_A, ρ_B). Denote by $\mathrm{Opext}(B, A)$ the set of equivalence classes of extensions (1.1) which give rise to the given "matching." Our main result is:

THEOREM 6.1. *There is a natural isomorphism:*

$$\overline{H}^3(B, A) = \mathrm{Opext}\,(B, A).$$

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References

1. M. Barr and J. Beck, *Homology and standard constructions*, Sem. on Triples and Categorical Homology Theory (ETH, Zurich, 1966/67) Springer, Berlin, 1969.

2. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Indust., no. 1252 = Publ. Math. Univ. Strasbourg, no. 13, Hermann, Paris, 1958. MR **21** #1583.

3. V. K. A. M. Gugenheim, *On extensions of algebras, co-algebras and Hopf algebras*. I, Amer. J. Math. **84** (1962), 349–382. MR **26** #1340.

4. J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math (2) **81** (1965), 211–264. MR **30** #4259.

5. W. Singer, *Extension theory for connected Hopf algebras*, Bull. Amer. Math. Soc. **76** (1970), 1095–1099.

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