HARMONIC MAPS; A DIRECT METHOD IN THE CALCULUS OF VARIATIONS

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Sampson and Eells [6] have shown the existence of harmonic maps in any homotopy class of maps from a compact Riemannian manifold with nonpositive sectional curvature. (An imbedding condition is necessary if the image manifold is not compact.) These results were extended by Hartman [2] to include a uniqueness result if the sectional curvature is negative. The original proofs of these existence and uniqueness theorems for harmonic maps, which are the solutions of nonlinear elliptic systems, rely on the properties of the related nonlinear parabolic equations. We present here a direct method, which uses a perturbation of the energy integral to an integral which can be shown to satisfy condition (C) of Palais and Smale. We then automatically get existence theorems for the new integrals and we show that the maps which minimize these new integrals converge to a minimizing function of the original integral. Regularity theorems for critical points seem to be essential for this method to work. The uniqueness theorem can be derived from Morse theory or directly from Ljusternik-Schnirelman theory.

This technique is a direct application of the abstract Ljusternik-Schnirelman theory for Banach manifolds due to Palais [5]. In many cases computations have been slighted and the reader is urged to refer to Sampson-Eells [6] or Hartman [2]. The author is indebted to J. J. Kohn, who suggested the problem. The fact that the critical points of J^λ in Theorem 2 (for $p=2$ and $\dim M<4$) are exactly the harmonic maps was observed originally by Eliasson [1].

Let M and N be C^∞ Riemannian manifolds without boundary, M compact and N complete. A map $s: M\to N$ is harmonic if s is a critical point of the integral $E(f)=\int_M |df|^2 d\mu$, where the norm on $df(x) \in T_x^*(M)\otimes T_{s(x)}N$ is in the induced metric.[1] The variational equation for harmonic maps is

$$0 = dE_f u = 2\int_M (df, \nabla_f u)d\mu = -2\int_M (\Delta f, u)d\mu$$

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[1] df indicates a section of the bundle $T(M)\otimes f^*T(N)$. ∇ will be the notation for the covariant differential in a bundle, or an operator from sections of η to $\eta\otimes T^*(M)$.

or $\Delta f=0$, where $\Delta f=\partial iv\, df=$contraction $\nabla_f df$. The covariant differential ∇_f on $T^*(M)\otimes f^*T(N)$ is induced by the covariant differential on $T^*(M)$ and $T(N)$, and the contraction is simply the inner product on $T^*(M)$.

All metrics, contractions, and covariant derivatives will be the naturally occurring ones. We also need the Banach manifolds

$$L_1^p(M, N)=L_1^p(M, R^l)\cap C^0(M, N), \quad L_2^q(M, N)=L_2^q(M, R^l)\cap C^0(M, N)$$

where $p=2q>\dim M$ and $N\subseteq R^l$ is a closed embedding. L_k^p denotes the Sobolev space of functions whose first k derivatives are p integrable. If a harmonic map is in $L_1^p(M, N)$ for $p>\dim M$, it is in $C^\infty(M, N)$, so it is immaterial in which space we look for harmonic maps. The following embedding condition is most practical for our purposes.

(1.1) The metric on N satisfies the embedding property if there exists a closed embedding of $N\subseteq R^l$ such that the metric on N is equivalent to the metric induced by this embedding, and if there exists a function $f(r)$, $\lim_{r\to\infty}f(r)=\infty$ such that for $x\in N\subseteq R^l$, the connected component of x in $N\cap D_xf(|x|)$ is contractable. $D_xf(|x|)$ is the ball in R^l about x with radius $f(|x|)$. This will mean that if we have a sequence of maps, $s_i: M\to N$, not homotopically trivial such that $\max_{x,y\in M}|s_i(x)-s_i(y)|_{R^l}$ is bounded, then $s_i(x)$ is contained in some fixed compact subset of N, for all $x\in M$ and all i.[2]

(1.2) We shall show existence by using condition (C) of Palais and Smale [4]. If L is a complete Finsler[3] manifold, a differentiable function J on L is said to satisfy condition (C) if for every set $S\subseteq L$ on which J is bounded and $|dJ|$ is not bounded away from zero, $\bar{S}$ contains a critical point of J. A function which is bounded below and satisfies condition (C) takes on its minimum in each component.

We shall prove the following two theorems:

THEOREM 1. *Let* $p>\dim M$.

$$E_\epsilon(f) = E(f) + \epsilon\int_M |df|^p d\mu = \int_M (|df|^2 + \epsilon|df|^p)d\mu.$$

(2.1) E_ϵ satisfies condition (C) on $L_1^p(M, N)$ if N is compact. If N is not compact but satisfies the embedding property (1.1), E_ϵ satisfies condition (C) on the nonhomotopically nontrivial components of $L_1^p(M, N)$. The critical points of E_ϵ are Lipschitz and in $L_2^2(M, N)$.

[2] This is not the embedding condition of Sampson and Eells.

[3] Function manifolds have a natural class of Finsler structures on them compatible with the metric of the domain and image spaces.

(2.2) If the sectional curvature of N is nonpositive, then the set of accumulation points of the minimizing functions $\{s_\epsilon\}$ of E_ϵ as $\epsilon \to 0$ is nonempty and consists of harmonic maps.

THEOREM 2. *If N has nonpositive sectional curvature and $p > \dim M$ and $\lambda > 0$,*

$$J^\lambda(f) = \int_M \lambda(1 + |\Delta f|^2)^{p/2} + |df|^2 d\mu$$

then:

(3.1) *J^λ satisfies condition* (C) *on $L_2^p(M, N)$ if N is compact. If N satisfies the embedding condition, J^λ satisfies condition* (C) *of the non-homotopically trivial components of $L_2^p(M, N)$. The critical points of J^λ are in $C^\infty(M, N)$.*

(3.2) *The critical points of J^λ are precisely the critical points of E.*

(3.3) *The critical sets are connected in each component of $L_2^p(M, N)$.*

(3.4) *If the sectional curvature of N is negative the critical sets are the manifolds:*

(a) *The maps of M to a point;*

(b) *Composition of a harmonic map from M to S^1 followed by a closed geodesic in N, and rotations;*

(c) *Weakly nondegenerate*[4] *minima.*

(3.5) *The critical manifolds in each component are unique and using Morse theory, the components of $L_2^p(M, N)$ can be retracted onto their harmonic maps.*

PROOF OF (2.1). For N compact, the fact that $\int_M |df|^p d\mu$ satisfies condition (C) is fairly well known [5], [9] and the proof for E_ϵ is similar. The embedding condition gives us the information that when $\int_M |df_i|^p d\mu$ is bounded, $\max_{x,y \in M} |f_i(x) - f_i(y)|_{R^l}$ is bounded and the images of $\{f_i\}$ lie in a compact subset, if the $\{f_i\}$ are not homotopically trivial, and the proof proceeds as for N compact. The fact that $|ds|$ is bounded if s is a critical point of E_ϵ in $L_1^p(M, N)$ follows from the estimates in Morrey [3 pp. 135–138].

PROOF OF (2.2). Let s_ϵ denote a minimum of E_ϵ in some fixed component of $L_1^p(M, N)$. $E(s_\epsilon)$ is a decreasing sequence. The variation of E_ϵ is

$$dE_{\epsilon,f}(u) = -2\int_M (\Delta f, u) d\mu - \epsilon p \int_M (\partial iv\, |df|^{p-2} df, u) d\mu,$$

[4] See reference [7].

$dE_{\epsilon,s}{}^{\epsilon}(u)=0$, and we may consider Δs_ϵ as a variation, as s_ϵ is just smooth enough.

$$0 = dE_{\epsilon,s_\epsilon}(\Delta s_\epsilon) = -2\int_M (\Delta s_\epsilon, \Delta s_\epsilon)d\mu$$

$$- \epsilon p \int_M (\partial iv \mid ds_\epsilon \mid^{p-2} ds_\epsilon, \partial iv\ ds_\epsilon)d\mu.$$

The first term is negative and we integrate the second term by parts (s_ϵ is not smooth enough but the equations are valid as this is true for smooth f) to get

$$0 \geqq \int_M (\partial iv \mid ds_\epsilon \mid^{p-2} ds_\epsilon, \partial iv\ ds_\epsilon)d\mu$$

$$= \int_M (\nabla \mid ds_\epsilon \mid^{p-2} ds_\epsilon, \nabla\ ds_\epsilon)d\mu$$

$$+ \int_M [-\overline{R}(s_\epsilon) + \overline{S}(s_\epsilon)] \mid ds_\epsilon \mid^{p-2} d\mu.$$

The terms from interchanging the order in differentiation are (in orthonormal coordinates x_i in M and s^α in N).

$$(\overline{R}(s_\epsilon))(x) = \sum_{i,j} \left((R(s_\epsilon(x)) \frac{\partial s_\epsilon}{\partial x_i}, \frac{\partial s_\epsilon}{\partial x_j}) \frac{\partial s_\epsilon}{\partial x_i}, \frac{\partial s_\epsilon}{\partial x_j} \right) < 0$$

where R is the Riemannian curvature tensor in N and

$$(S(s_\epsilon))x = \sum_\alpha S(x)(d\overset{\alpha}{s_\epsilon}(x), d\overset{\alpha}{s_\epsilon}(x))$$

where S is the Ricci curvature of M.

We consequently can show that:

$$\int_M \mid d(\mid ds_\epsilon \mid^{p/2}) \mid^2 d\mu \leqq S \int_M \mid ds_\epsilon \mid^p d\mu$$

where S depends on the Ricci curvature of M and is zero if this tensor is nonnegative. Since we know $\int_M |ds_\epsilon|^2 d\mu$ is bounded, we can use Sobolev and interpolation inequalities to conclude that $\int_M |ds_\epsilon|^p d\mu$ is bounded. We choose a subsequence $\{s_{\epsilon(i)}\}$ which converges in C^α, $\alpha < p/n - 1$, and this sequence will converge to a minimum of E in $L_1^p(M, N)$. We make strong use of the Sobolev imbedding theorems as well as interpolation properties.

PROOF OF (3.1). The fact that J^λ satisfies condition (C) can be easily shown once we have shown the property: If $J^\lambda(f)$ is bounded on a set, then $\int_M |df|^p d\mu$ is bounded. We can then select a subsequence convergent in $C^0(M, N)$ and local estimates can be made [5]. But

$$C' \int_M |d|df|^{p/2}|^2 d\mu \leqq \int_M (\nabla |df|^{p-2} df, \nabla df) d\mu$$

$$\leqq \int_M (\partial iv |df|^{p-2} df, \Delta f) + (\overline{R}(f) - \overline{S}(f)) |df|^{p-2} d\mu$$

$$\leqq \epsilon \int_M |d|df|^{p/2}|^2 d\mu + C \int_M |df|^p d\mu + C(\epsilon) \int_M |\Delta f|^p d\mu$$

and by the identical arguments used in the proof of (2.2), if

$$J^\lambda(f) = \lambda \int_M (1 + |\Delta f|^2)^{p/2} d\mu + E(f)$$

is bounded, then $\int_M |df|^p d\mu$ is bounded. The fact that the critical points are in C_2^α for $\alpha > 0$ is proved in [8] and it follows from usual regularity theorems that they are C^∞.

PROOF OF 3.2. The variation of J^λ at f can be seen to be

$$dJ_f^\lambda(u) = \lambda p \int_M (1 + |\Delta f|^2)^{p/2} \cdot (\Delta f, \Delta u + R(f(x))(df, u) df) d\mu - 2 \int_M (\Delta f, u) d\mu.$$

The term with curvature is to be contracted using the metric in $T^*(M)$. Clearly if $\Delta s = 0$, then $dJ_s^\lambda(u) = 0$. If $dJ_s^\lambda = 0$, we may use the variation $u = \Delta s$, since it is smooth, which implies, after the first term is integrated by parts, that each term is nonpositive and therefore zero.

Statement (3.3) can be proved using Ljusternik-Schnirelman category theory. Statement (3.4) is proved by computing the second-variation of E at a harmonic function.

$$H(u, v) = d^2 E_s(u, v) = \int_M (\nabla u, \nabla v) - (R(s(x))(ds, u) ds, v) d\mu$$

where the curvature term is again contracted. $H(u, u) > 0$ unless

$\nabla u=0$ and $\partial s/\partial x_i=g_i u$ if $u\neq 0$, in the case that the curvature is actually negative. $d^2J_s^\lambda(u, u)\geqq H(u, u)$ and is in fact equal to zero when $H(u, u)=0$. Clearly (a) and (c) occur. If there exists $u\neq 0$ such that $H(u, u)=0$ and $ds\neq 0$, the image of s must be one-dimensional in N and some computation shows it is a geodesic.

Statement (3.5) is easy to show if the minimum is an isolated critical point, because it will then be weakly nondegenerate [7]. However, the critical manifolds are weakly nondegenerate minima, and the theory is easy to extend. A special argument is necessary in the trivial component of $L_2^p(M, N)$ if N is not compact, since condition (C) is not satisfied in this case.

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