BEST UNIFORM APPROXIMATIONS VIA ANNIHILATING MEASURES

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The problem under consideration in this paper is that of uniformly approximating an arbitrary continuous function g on the closed unit disk $\bar{D}$ by continuous functions f which are analytic in $D=\{z \text{ complex}: |z|<1\}$. In particular, we are concerned with the existence, uniqueness, and construction of a best approximation f_0 to g. Our results consist of a proof of the uniqueness of f_0 when it exists and an algorithm for constructing f_0 for certain classes of functions g. Both results follow from a more general theorem on best uniform approximations and annihilating measures.

If E is a normed linear space, A is a subspace of E, and S_A^* consists of all the linear functionals L on E with $\|L\| \leqq 1$ and which vanish on A then, as a consequence of the Hahn-Banach theorem, the following relationship holds [1].

THEOREM 1. *If $g \in E$ then*

$$\|g\|_A = \inf_{f \in A} \|g-f\| = \max_{L \in S_A^*} |L(g)|.$$

For $E=C(K)$, the continuous complex valued functions defined on the compact Hausdorff space K, additional information can be obtained from Theorem 1 by applying the Riesz representation theorem [4] to $L \in S_A^*$. Here $\|g\| = \max_{z \in K} |g(z)|$ is the uniform norm.

THEOREM 2. *If $g \in C(K)$, $f_0 \in A$ is a best uniform approximation to g, $L \in S_A^*$, and $L(g)=\|g\|_A$ then $g-f_0=\|g\|_A \bar{\phi}$ a.e. $d\mu$ where $\phi d\mu$ is the polar decomposition of the unique regular Borel measure on K which represents L.*

PROOF. By Theorem 1, there is an $L \in S_A^*$ with $L(g)=\|g\|_A$ and $\|L\|=1$. Let $\phi d\mu$ be the measure which represents L where $|\phi|=1$ a.e. $d\mu$, $d\mu \geqq 0$ and $\int_K d\mu=1$. Now,

$$\|g\|_A = \int_K (g-f_0)\phi d\mu \leqq \int_K |(g-f_0)\phi|\, d\mu \leqq \int_K \|g-f_0\| d\mu = \|g\|_A.$$

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Therefore,

$$\int_K (g - f_0)\phi d\mu = \int_K |(g - f_0)\phi| \, d\mu = \|g\|_A.$$

Since $|(g-f_0)| \leqq \|g\|_A$ we must have $|g-f_0| = \|g\|_A$ on the support of $d\mu$.

Then it follows that

$$(g - f_0)\phi = \|g\|_A \text{ a.e. } d\mu$$

and

$$g - f_0 = \|g\|_A \bar{\phi} \text{ a.e. } d\mu$$

which was to be proved.

Let $K = \bar{D}$, the closed unit disk, and let A consist of all functions in $C(\bar{D})$ which are analytic in D. Then the support of the measure $d\mu$ in Theorem 2 is large enough to ensure the uniqueness of f_0 when f_0 exists.

THEOREM 3. *If $f_0 \in A$ is a best uniform approximation to $g \in C(\bar{D})$ then f_0 is unique.*

PROOF. Suppose f_0 is not unique. Then there is an $f_1 \in A$, $f_1 \neq f_0$ such that $\|g-f_0\| = \|g-f_1\| = \|g\|_A$. Let $\phi d\mu \in S_A^*$ be the measure in Theorem 2. Then $f_0 = f_1 = g - \|g\|_A \bar{\phi}$ a.e. $d\mu$ and $h = f_1 - f_0 = 0$ on K, the support of μ. Therefore $K \cap D$ can not have a limit point in D so $K \cap D$ is at most countable.

One can now show that A is dense in $L^2(d\mu, K\cap D)$ and therefore $\phi d\mu$ is the zero measure on $K \cap D$. Then, by the F. and M. Riesz theorem [3], $\phi d\mu$ is absolutely continuous with respect to Lebesgue measure on ∂D. But $f_0 = f_1$ on $\bar{D}$ if $K \cap \partial D$ has positive Lebesgue measure [3]. Therefore, $K \cap \partial D$ has zero Lebesgue measure and $g \in A$ so that $g = f_0 = f_1$ which contradicts our assumption.

We now demonstrate the existence of best approximations f_0 in A to harmonic functions of the form $g = \sum_{j=0}^n b_j \bar{z}^{j+1}$. The maximum modulus principle for harmonic functions implies that we may restrict our attention to approximating g on ∂D by f in A. The linear functionals that annihilate A on ∂D are of the form

$$L(h) = \int_0^{2\pi} h(e^{i\theta}) e^{i\theta} f(e^{i\theta}) \frac{d\theta}{2\pi}$$

where $f \in H^1$, the Hardy space H^1, i.e., f is an L^1 function on ∂D whose harmonic extension to D is analytic. We will simply write $L(h)$

$=\int zhf\,(d\theta/2\pi)$ for the above integral. Theorem 1 then says

$$(1)\qquad \|g\|_A = \max_{f\in H^1} \frac{\left|\int zgf\,(d\theta/2\pi)\right|}{\|f\|_1}.$$

However, $\|g\|_A$ may be calculated by considering a much smaller class of linear functionals.

LEMMA. *If* $g=\sum_{j=0}^n b_j\bar{z}^{j+1}$ *then*

$$\|g\|_A = \max_{f \text{ outer}; f\in P_n} \frac{|(f^2,\bar{z}\bar{g})|}{(f,f)}$$

where $(f^2,\bar{z}\bar{g})$ *and* (f,f) *are inner products in* $L^2(d\theta/2\pi,\partial D)$.

PROOF. Rudin and de Leeuw have shown that if $f\in H^1$ and $\|f\|_1=1$ then $f=\frac{1}{2}(f_1+f_2)$ where f_1 and f_2 are outer functions in H^1 with L_1-norms$=1$ [2]. Hence (1) reduces to

$$\|g\|_A = \max_{f \text{ outer}; f\in H^1} \frac{|(f,\bar{z}\bar{g})|}{\|f\|_1}.$$

But f being outer in H^1 implies that $f^{1/2}$ is outer in H^2. Therefore,

$$\|g\|_A = \max_{f \text{ outer}; f\in H^2} \frac{|(f^2,\bar{z}\bar{g})|}{(f,f)}.$$

Let P_n denote the space of all polynomials in z of degree $\leqq n$ and f_n denote the $L^2(d\theta/2\pi)$ projection of f onto P_n. Then

$$\frac{|(f^2,\bar{z}\bar{g})|}{(f,f)} \leqq \frac{|(f_n^2,\bar{z}\bar{g})|}{(f_n,f_n)}$$

since the numerators are equal and $(f,f)\geqq(f_n,f_n)$. Since the inequality is strict for $f\notin P_n$ we have

$$\|g\|_A = \max_{f \text{ outer}; f\in P_n} \frac{|(f^2,\bar{z}\bar{g})|}{(f,f)}.$$

THEOREM 4. *If* $g=\sum_{j=0}^n b_j\bar{z}^{j+1}$ *then there is a rational function* f_0 *in* A *which is the unique best approximation to* g.

PROOF. Applying the lemma, there is an f in P_n, f outer, with $(f,f)=1$ and $\|g\|_A=\int zgf^2\,(d\theta/2\pi)$. The polar decomposition of the measure is $(zf/\bar{f})|f|^2\,(d\theta/2\pi)$ where $\phi=zf/\bar{f}$ and $d\mu=|f|^2\,(d\theta/2\pi)$. On ∂D, $\bar{\phi}=\bar{z}\bar{f}/f$ has at most n discontinuities which are removable so let $\bar{\phi}$

denote the modification of $\bar{z}\bar{f}/f$ which is continuous on ∂D. We claim that $f_0 = g - \|g\|_A \bar{\Phi}$ is the unique best approximation to g from A.

To show that $f_0 \in A$ consider a sequence $f_n \in A$ with $\|g - f_n\| \to \|g\|_A$. Then by either a normal family argument on $\{f_n\}$ or a weak* compactness argument [2] there is an $h \in H^\infty$ with $\|g-h\|_\infty = \|g\|_A$. Applying the proof of Theorem 2 we have $h = f_0$ a.e. $d\theta$ on ∂D. Consequently both functions have the same analytic extension to D and hence $f_0 \in A$.

Uniqueness of f_0 follows from Theorem 3 and the fact that f_0 is a rational function follows from our algorithm for calculating f_0 which we describe next.

Let g, f, and f_0 be as above where $f = \sum_{j=0}^n a_j z^j$, $\rho = \|g\|_A$, $(f, f) = 1$, and $\rho = (f^2, \bar{z}\bar{g})$. Then $(g - f_0)\, zf/\bar{f} = \rho$ a.e. $d\theta$ or

(2) $zgf - zf_0 f = \rho \bar{f}$ on ∂D.

The nonpositive Fourier coefficients in (2) satisfy the matrix equation

(3) $BF = \rho \bar{F}$ where

$$B = \begin{bmatrix} b_0 & b_1 \cdots & b_n \\ b_1 & \cdots b_n & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_n & 0 \cdots & 0 \end{bmatrix}, \quad F = \begin{bmatrix} a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \quad \text{and} \quad \bar{F} = \begin{bmatrix} \bar{a}_0 \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_n \end{bmatrix}.$$

From (3) can be derived

(4) $(\bar{B}B - \rho^2 I)\, F = 0$.

The method for finding f_0 consists of first finding the largest positive eigenvalue ρ^2 of $\bar{B}B$ and then solving $(\bar{B}B - \rho^2 I)X = 0$ for a nontrivial solution $X = F_1$. Then either F_1 or $(i)(\bar{B}\bar{F}_1 - \rho F_1)$ is a nontrivial solution of $B\bar{X} = \rho X$. Let F denote that nontrivial solution. Then f is defined by F and f_0 by $f_0 = g - \rho\, \bar{z}\bar{f}/f$ on ∂D. Choosing F so that $(F, F) = 1$ shows that $\rho^2 = \|g\|_A^2$ is the largest positive eigenvalue of $\bar{B}B$ since F is a solution to both (3) and (4) and since $\int zgf^2\,(d\theta/2\pi) = F^t BF = \rho$, where F^t is the transpose of F.

As an example of the preceding method consider $g = 3\bar{z} + 2\bar{z}^2$. Then

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \bar{B}B = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}.$$

$\text{Det}(\bar{B}B - \rho^2 I) = (\rho^2 - 16)(\rho^2 - 1)$. Hence $\|g\|_A = 4$. Now $F = \binom{2}{1}$ is a solution of (3). Therefore, let $f = 2 + z$. Then

$$f_0 = g - 4\bar{\Phi} = \frac{3z + 2}{z^2} - \frac{4}{z^2}\left(\frac{1 + 2z}{z + 2}\right) \quad \text{on } \partial D.$$

Thus $f_0=3/(z+2)$ is the unique best approximation from A to g on $\overline{D}$.

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