COMPLETELY REGULAR MAPPINGS AND DIMENSION[1]

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1. **Introduction.** In an earlier paper [12] the author proved the following theorem: There exists a monotone open map of the universal curve onto any continuous curve such that each point-inverse set is also a universal curve. Since these mappings are open and have homeomorphic point-inverse sets, it is natural to ask whether or not these mappings are completely regular. Theorem 1 of this paper shows that they will be completely regular only if the range is a point. Theorem 1, Theorem 3, and the corollary to Theorem 3 all give conditions on completely regular mappings so that they will not raise dimension. Theorem 4 actually classifies completely regular mappings of a certain type.

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2. **The main theorem.**

THEOREM 1. *If f is a completely regular mapping of an n-dimensional compactum X onto a compactum Y and $\check{H}^n(f^{-1}(y)) \neq 0$ for all $y \in Y$, then Y is 0-dimensional.*

LEMMA 1. *Let X be an n-dimensional compactum. Let J be a finite polyhedron contained in E^{2n+1} of dimension less than $n+1$. If f is a mapping of X into E^{2n+1} and $\eta > 0$, then there exists a homeomorphism $h: X \to E^{2n+1}$ such that $d(f, h) < \eta$ and $h(X) \cap J = \emptyset$.*

PROOF OF LEMMA 1. Approximate f by a mapping g whose range is contained in an n-polyhedron which (by general positioning) misses J. Since the set of homeomorphisms is dense in the function space $(E^{2n+1})^X$, we can find a homeomorphism h which approximates g and such that $h(X) \cap J = \emptyset$.

The homology theory in this paper will be singular homology with integer coefficients. If J is a singular n-cycle, then $|J|$ will denote its

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carrier. The cohomology theory will be Čech cohomology with integer coefficients.

LEMMA 2. *Let M and N be compact subsets of E^{2n+1} and let J be an n-cycle which represents a class in $H_n(E^{2n+1}-M)$. If there exists a homotopy equivalence h of M into N which moves no point of M more than $\frac{1}{2}d(M, |J|)$, then J represents a nonzero class in $H_n(E^{2n+1}-M)$ if and only if J represents a nonzero class in $H_n(E^{2n+1}-N)$.*

PROOF. Let U be an open subset of E^{2n+1} such that $|J| \subseteq U \subseteq E^{2n+1}-(M\cup N)$, $E^{2n+1}-U$ is compact, and $d(U, M) > \frac{1}{2}d(M, |J|)$.

If i_1 and i_2 denote inclusion mappings, then the above restrictions insure that the following diagram is homotopy commutative:

$$\begin{array}{ccc} M & \xrightarrow{\;i_1\;} & E^{2n+1} - U \\ \downarrow h & \nearrow i_2 & \\ N & & \end{array}$$

Thus, $i_1^* = h^* \circ i_2^*$.

From the naturality of Alexander Duality, we get the following commutative diagram:

$$\begin{array}{ccc} H_n(E^{2n+1} - M) & \xrightarrow{\approx} & \check{H}^n(M) \\ \uparrow i_* & & \uparrow i_1^* \\ H_n(U) & \xrightarrow{\approx} & \check{H}^n(E^{2n+1} - U) \\ \downarrow i_* & & \downarrow i_2^* \\ H_n(E^{2n+1} - N) & \xrightarrow{\approx} & \check{H}^n(N) \end{array}$$

Since $i_1^* = h^* \circ i_2^*$ and h^* is an isomorphism, the result follows.

LEMMA 3. *Let A and B be disjoint closed subsets of an n-dimensional compactum X. If $\check{H}^n(A) \neq 0$, then there exists an imbedding $h: X \to E^{2n+1}$ and n-cycle J such that $h(X) \cap |J| = \varnothing$ and J represents a nonzero class in $H_n(E^{2n+1}-h(A))$ and the zero class in $H_n(E^{2n+1}-h(B))$.*

PROOF. Let f_1 be an imbedding of A into E^{2n+1} and let J be a simplicial n-cycle which represents a nonzero class in $H_n(E^{2n+1}-f_1(A))$. By Tietze's extension theorem, we can find a map f from X into E^{2n+1} which extends f_1 and such that $f(B)$ is outside some ball N containing $f_1(A)\cup|J|$. By Lemma 1 there exists a homeomorphism h of X into E^{2n+1} such that

$$d(f, h) < \min\{d(f(B), N), \tfrac{1}{2}d(|J|, f(A))\}$$

and $h(X)\cap|J|=\varnothing$. By Lemma 2, J represents a nonzero class in $H_n(E^{2n+1}-h(A))$.

Note that in the proof of Lemma 3 we did not try to extend the imbedding f_1 to h, but rather "moved" it slightly. This is neceIsary since the examples in [2] can be used to show that there exists an imbedding of a Cantor set plus a circle into E^n, $n\geqq 5$, which cannot be extended to a Cantor set plus a disk.

Proof of Theorem 1. Since Y is compact, it is sufficient to show that each component K of Y is a point.

Suppose K contains two distinct points y_1 and y_2. By Lemma 3 we can find an imbedding h of X into E^{2n+1} and n-cycle J such that $h(X)\cap|J|=\varnothing$ and the class $[J]$ is nonzero in $H_n(E^{2n+1}-h(f^{-1}(y_1)))$ and $[J]=0$ in $H_n(E^{2n+1}-h(f^{-1}(y_2)))$.

Let $Y_1=\{y\in K: [J]\neq 0 \text{ in } H_n(E^{2n+1}-h(f^{-1}(y)))\}$ and $Y_2=\{y\in K: [J]=0 \text{ in } H_n(E^{2n+1}-h(f^{-1}(y)))\}$. Note that $y_1\in Y_1$ and $y_2\in Y_2$. Using Lemma 2 together with the complete regularity of f it is easy to show that both Y_1 and Y_2 are open. Since $K=Y_1\cup Y_2$, we have a contradiction of the assumption that K is connected. Thus, K is a point.

Remark. Note in Theorem 1 that if Y is connected, then Y is a point.

3. Mappings which do not raise dimension.

The next theorem is a cohomology analogue of Theorem 1 of [4].

Theorem 2. *If f is a monotone mapping of an n-dimensional compactum X onto a finite dimensional compactum Y and $H^k(f^{-1}(y))=0$ for all $k\geqq 1$ and all $y\in Y$, then*

$$\dim Y \leqq n \leqq \dim Y + \sup\{\dim f^{-1}(y)\}.$$

Proof. The second inequality follows from Theorem VI of [8].

To show the first inequality we will use the characterization of dimension given in Theorem VIII 2 of [10]. Thus, it is sufficient to show that if C is a closed subset of Y and $m\geqq n$, then the homomorphism i^*, induced by the inclusion $i: C\to Y$, is onto. Since $\dim X=n$, we have the following commutative diagram:

$$\begin{array}{ccccc} 0 \leftarrow & \check{H}^m(f^{-1}(C)) & \leftarrow & \check{H}^m(X) \\ & \uparrow f^* & & \uparrow f^* \\ & \check{H}^m(C) & \xleftarrow{i^*} & \check{H}^m(Y) \end{array}$$

By the Vietoris Mapping Theorem we know that f^* is an isomorphism so that i^* is onto. Thus, dim $Y \leqq n$.

THEOREM 3. *Let f be a completely regular mapping of an n-dimensional compactum X onto a finite dimensional compactum Y. If for each $y \in Y$ $f^{-1}(y)$ has only a finite number of components and $\check{H}^k(f^{-1}(y)) = 0$ for $k = 1, \cdots, n-1$, then* dim $Y \leqq$ dim X.

PROOF. It is sufficient to show that each component K of Y has dimension $\leqq$ dim X.

If $\check{H}^n(f^{-1}(y)) \neq 0$ for some $y \in K$, then, by Theorem 1, K is a point.

If $\check{H}^n(f^{-1}(y)) = 0$ for all $y \in K$, then factor f by the monotone light factorization theorem:

$$\begin{array}{ccc} f^{-1}(K) & \xrightarrow{\;f\;} & K \\ {\scriptstyle m}\Big\downarrow & \nearrow{\scriptstyle l} & \\ Z & & \end{array}$$

Since l is a finite to one open mapping, we know that dim Z $=$ dim $K < \infty$ [9]. Hence, by applying Theorem 2 to the map m, we know that dim $K =$ dim $Z \leqq$ dim X.

COROLLARY. *If f is a completely regular mapping of a 1-dimensional compactum X onto a finite dimensional compactum Y such that each point-inverse has a finite number of components, then* dim $Y \leqq 1$.

REMARK. Note that the above corollary is not true if we allow the point-inverse sets to be cantor sets. For Theorem 2 of [12] states that there exists a light open mapping of the universal curve onto any nondegenerate continuous curve such that each point-inverse set is a Cantor set. Note that these mappings will be completely regular.

EXAMPLE. If Y is any continuous curve, then there exists a 2-dimensional continuum X and a monotone completely regular mapping F of X onto Y.

Let σ^7 denote the standard 7-simplex. If $A, B \subset \sigma^7$, then let $A \cdot B = \{x \in \sigma^7 : x \in \langle a, b \rangle, a \in A, b \in B\}$. Let σ_1^3 and σ_2^3 denote two disjoint 3-simplices which are faces of σ^7. Let u_1 and u_2 be two copies of the universal curve in σ_1^3 and σ_2^3, respectively. Let $f_i: U_i \to Y$ be completely regular mappings of the type in the above remark. Let $X = \{f_1^{-1}(y) \cdot f_2^{-1}(y) : y \in Y\}$. There is a natural monotone completely regular mapping F of X onto Y which extends each f_i. It can be shown that dim $F^{-1}(y)=1$ and dim $X=2$. Note that this example follows the technique of [7].

THEOREM 4. *If f is a completely regular mapping of a 1-dimensional compactum X onto a finite dimensional continuum Y such that $f^{-1}(y)$ is a continuous curve for all $y \in Y$, then either Y is a point or Y is homeomorphic to X under f.*

PROOF. If $f^{-1}(y)$ contains a simple closed curve, then by Theorem 1 we know that Y is a point.

Suppose $f^{-1}(y)$ contains no simple closed curve for all $y \in Y$. If $f^{-1}(y)$ and Y are both nondegenerate, then by Theorem 3 of [4] we know that dim $X=$ dim $Y+1 \geqq 2$. This contradicts the assumption that X is 1-dimensional.

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