DUALITY OF MULTIPLICATIVE FUNCTIONALS

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1. **Introduction.** Suppose X and $\hat{X}$ is a pair of standard processes in duality relative to a Radon measure ξ. We refer the reader to [1] for all terminology and notation not explicitly defined here. In particular (U^α) and $(\hat{U}^\alpha)$ denote the resolvents of X and $\hat{X}$ respectively and the α-potential kernel $u^\alpha(x, y)$ satisfies

$$U^\alpha(x, dy) = u^\alpha(x, y)dy, \qquad \hat{U}^\alpha(x, dy) = u^\alpha(y, x)dy.$$

Here $dy = \xi(dy)$. We make no regularity assumptions on the resolvents of X and $\hat{X}$. One of the most important properties of such dual processes is (VI-1.16) (all such references are to [1]) which states that if A is a Borel set then for all $\alpha \geqq 0$ and x, y

$$P_A^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_A^\alpha(x, y). \tag{1.1}$$

This result which is due to Hunt says that the process X killed at the time it first hits A and the process $\hat{X}$ killed when it first hits A are in duality. In particular if we define

$$Q_t f(x) = E^x\{f(X_t); t < T_A\} \quad \text{and} \quad \hat{Q}_t f(x) = \hat{E}^x\{f(X_t); t < T_A\}$$

(for typographical reasons we will omit the hat "^" in those places where it is obviously required—see the remark on p. 262 of [1]), then it is a standard observation that (1.1) is equivalent to

$$(Q_t f, g) = (f, \hat{Q}_t g) \tag{1.2}$$

for all $t \geqq 0$ and for all continuous functions with compact support, f and g. Here $(\phi, \psi) = \int \phi(x)\psi(x)dx$.

The purpose of this paper is to announce an extension of (1.2) and (1.1) to a more general class of multiplicative functionals than those of the form $M_t = I_{[0,T_A)}(t)$. Our basic result is that if M is an exact MF (multiplicative functional) of X then there exists a unique exact MF, $\hat{M}$, of $\hat{X}$ such that (1.2) holds where $\{Q_t\}$ and $\{\hat{Q}_t\}$ are the semigroups generated by M and $\hat{M}$ respectively and that an appropriate

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analogue of (1.1) also holds. Actually the existence of such an $\hat{M}$ is an easy consequence of a result of Meyer [3] and undoubtedly is known to many people. Our key result is the fact that this correspondence is multiplicative, that is $(MN)\hat{} = \hat{M}\hat{N}$, and it is this fact that turns the above correspondence into a useful tool. In particular this gives a new proof of some recent results of Revuz [4]. Detailed proofs and applications will appear elsewhere.

2. **Description of results.** Let $M=(M_t)$ be an MF of X; throughout this paper all MF's are assumed to be right continuous, to satisfy $0 \leqq M_t \leqq 1$, and to vanish on the interval $[\zeta, \infty]$. Moreover equality between MF's will always mean equivalence. See [1, III-1.6].

Let $Q_t f(x) = E^x\{f(X_t)M_t\}$ for $t \geqq 0$ and

$$V^\alpha f(x) = E^x\left\{\int_0^\infty e^{-\alpha t} f(X_t) M_t dt\right\}$$

for $\alpha \geqq 0$, so that (Q_t) and (V^α) denote the semigroup and resolvent generated by M. For each $\alpha \geqq 0$, define

$$\begin{aligned} P_M^\alpha f(x) &= -\, E^x\left\{\int_0^\infty e^{-\alpha t} f(X_t) dM_t\right\} \qquad && \text{if } x \in E_M, \\ &= f(x) && \text{if } x \notin E_M. \end{aligned}$$

Here $E_M = \{x : P^x(M_0 = 1) = 1\}$ is the set of permanent points of M. It is well known and easy to check that, at least for $\alpha > 0$,

$$U^\alpha - V^\alpha = P_M^\alpha U^\alpha. \tag{2.1}$$

From here on we assume that X and $\hat{X}$ are standard processes in duality relative to a Radon measure $\xi(dx) = dx$. Then using standard techniques one obtains a function $v^\alpha(x, y)$ such that $V^\alpha f(x) = \int v^\alpha(x, y) f(y) dy$ and

$$u^\alpha(x, y) = v^\alpha(x, y) + P_M^\alpha u^\alpha(x, y). \tag{2.2}$$

If we now define $\hat{V}^\alpha f(x) = \int v^\alpha(y, x) f(y) dy$, it is easy to check, see Meyer [3], that $(\hat{V}^\alpha)$ is a resolvent *exactly* subordinate to $(\hat{U}^\alpha)$. Consequently it follows from results of Meyer, [3] and [1, III-2.3], that there exists an exact MF, $\hat{M}$, of $\hat{X}$ which generates $(\hat{V}^\alpha)$. We will write $\hat{P}_M^\alpha$ instead of $\hat{P}_{\hat{M}}^\alpha$ for the operator associated with $\hat{M}$, and $\hat{P}_M^\alpha(dy, x)$ for the corresponding measure. This discussion leads to the following theorem.

(2.3) THEOREM. *If M is an exact MF of X, then there exists a unique exact MF, $\hat{M}$, of $\hat{X}$ such that*

$$(2.4) \qquad P_M^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_M^\alpha(x, y),$$

which is equivalent to $(V^\alpha f, g) = (f, \hat{V}^\alpha g)$ *for all* $\alpha > 0$ *and* $f, g \in C_K^+$. *Moreover the mapping* $M \to \hat{M}$ *is bijective (from the class of exact MF's of X to the class of exact MF's of* $\hat{X}$*). If* $\hat{E}_M$ *is the set of permanent points of* $\hat{M}$*, then* $E_M \Delta \hat{E}_M$ *is semipolar. Also* $E_M - \hat{E}_M$ *is polar relative to* (X, M)*, and so if M does not vanish on* $[0, \zeta)$ *then* $E - \hat{E}_M$ *is polar.*

(2.5) THEOREM. *The map* $M \to \hat{M}$ *is multiplicative in the sense that if M and N are exact MF's of X, then* $(MN)\hat{} = \hat{M}\hat{N}$.

(2.6) COROLLARY. *If T is an exact terminal time for X, then there exists a unique exact terminal time* $\hat{T}$ *for* $\hat{X}$ *such that for all* $\alpha \geq 0$, $P_T^\alpha u^\alpha(x, y) = u^\alpha \hat{P}_T^\alpha(x, y)$.

It follows from (1.1) that if A is a Borel set and $T = T_A$ then $\hat{T} = \hat{T}_A$. It is also fairly easy to check that if h is a bounded nonnegative Borel function and $M_t = \exp(-\int_0^t h(X_s)ds)$, then $\hat{M}_t = \exp(-\int_0^t h(\hat{X}_s)ds)$. Combining these remarks with (2.5) and using an easy passage to the limit one obtains the full strength of the duality relationships proved by Hunt [2].

Let $S = \inf\{t: M_t = 0\}$ and $\hat{S} = \inf\{t: \hat{M}_t = 0\}$. Then S and $\hat{S}$ are dual terminal times, although they need not be exact. We will say that M is *continuous* provided $t \to M_t$ is continuous on $[0, S)$ almost surely, and that M is *natural* provided $t \to M_t$ and $t \to X_t$ have no common discontinuities on $[0, S)$ almost surely. With these definitions we have the following theorem.

(2.7) THEOREM. *If M is continuous, then* $\hat{M}$ *is continuous. If M is natural, then* $\hat{M}$ *is natural.*

The following corollaries are closely related to some recent results of Revuz [4].

(2.8) COROLLARY. *Let A be a continuous additive function of X that is finite on* $[0, \zeta)$ *almost surely, and let* $M_t = \exp[-A_t]$. *Then there is a unique continuous additive functional* $\hat{A}$ *of* $\hat{X}$ *restricted to* $\hat{E}_M$ ($E - E_M$ *is polar in this case) such that* $(f, U_A^\alpha V^\alpha g) = (\hat{U}_{\hat{A}}^\alpha \hat{V}^\alpha f, g)$.

In [4] Revuz associates a measure ν_A with any additive functional A.

(2.9) COROLLARY. *Let A be as in* (2.8). *Then* $\nu_A = \nu_{\hat{A}}$.

It is known from Revuz's work that ν_A is σ-finite and does not charge semipolar sets for A as above. If, in addition, A has a finite

α-potential, then $U_A^\alpha(x, dy) = u^\alpha(x, y)\nu_A(dy)$ for all x and $\hat{U}_A^\alpha(x, dy) = \nu_A(dy)u^\alpha(y, x)$ for $x \in \hat{E}_M$. These last results can be extended to natural additive functionals under some additional restrictions.

REFERENCES

1. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Academic Press, New York, 1968.
2. G. A. Hunt, *Markoff processes and potentials*. III, Illinois J. Math. **2** (1958), 151–213. MR **21** #5824.
3. P. A. Meyer, *Semi-groupes en dualité*, Séminaire de Théorie du Potentiel, 1960/61, Faculté des Sciences de Paris, Secrétariat mathématique, Paris, 1961. MR **28** #2031.
4. D. Revuz, *Mesures associées aux fonctionnelles additives de Markov*, Trans. Amer. Math. Soc. **148** (1970), 501–531.

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