ON CONTINUITY AND SMOOTHNESS OF GROUP ACTIONS

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Communicated by Armand Borel, February 20, 1970

In this note we give a short proof of a theorem of Bochner and Montgomery [1] using semigroup theory. In addition, we obtain more general results and give some applications (to diffeomorphism groups and nonlinear semigroups). A more detailed exposition will appear in [3].

1. Separate and joint continuity of group actions. The following generalizes a result of Ellis [6].

THEOREM 1. *Let M be a metric space, and let G be a Baire space with a group structure in which multiplication is separately continuous. Let $\pi: G\times M\to M$ be an action which is separately continuous. Then π is jointly continuous.*

PROOF. For each $x\in M$ there is a dense $\mathcal{G}_\delta$ $A\subset G$ such that π is continuous at (g_0, x) for $g_0\in A$. (See [4, p. 256, Problem 11] or [2, p. 255 Exercise 23].) For any $(g, x)\in G\times M$, we have, writing $\pi(g', x') = g'x'$,

$$g'x' = gg_0^{-1}g_0g^{-1}g'x' = \phi(g_0g^{-1}g'x')$$

where $\phi: M\to M$, $\phi(y) = gg_0^{-1}y$ and is continuous. But $g_0g^{-1}g'\to g_0$ as $g'\to g$, so as $g'\to g$, $x'\to x$ we get

$$g'x' = \phi(g_0g^{-1}g'x') \to \phi(g_0x) = gx$$

by joint continuity of π at (g_0, x).

If M is not metric the conclusion of Theorem 1 is no longer valid (consider G the circle and M the continuous functions on G with the topology of pointwise convergence).

As a corollary, it is not hard to deduce that a Baire separable metric group G for which multiplication is continuous is a topological group (compare [9] and [2, p. 258]). For example the C^k or H^s or $C^{k+\alpha}$ diffeomorphism group of a compact manifold satisfies these conditions; c.f. [5].

AMS 1969 *subject classifications.* Primary 2240, 4750, 4780; Secondary 4638.

Key words and phrases. Separate and joint continuity, manifold, nonlinear one parameter group, semigroup, diffeomorphism groups, Stone's theorem.

THEOREM 2. *Let G be a Baire group and M a (paracompact) manifold (finite or infinite dimensional). Let $\pi: G\times M\to M$ be separately (hence jointly) continuous. Suppose that for each $g\in G$, the map $\pi_g: x\mapsto gx$ is of class C^1 on M. Then the tangent action $T\pi: G\times TM\to TM$ is continuous.*

The idea is as follows. By Theorem 1 it suffices to show separate continuity of $T\pi(g, v_x)$ for $v_x\in T_xM$. Continuity in v_x is the hypothesis. But $\psi(g)=T\pi_g(v_x)$ is the pointwise limit (in a chart) of

$$\psi_n(g) = (g\cdot x, n[\pi_g(x+(1/n)v_x) - \pi_g(x)])$$

so by Baire's theorem, ψ has points of continuity. We get continuity at all g by using the group property as in Theorem 1.

There is an analogous theorem for $T^k\pi$ if π_g is C^k.

An example on $\boldsymbol{R}$ shows that in Theorem 2, mere differentiability of π_g will not suffice.

2. **A uniqueness theorem for nonlinear one parameter groups.** Theorem 2 can be used to prove the following:

THEOREM 3. *Let M be a manifold and $D\subset M$ a dense subset. Let X be a vector field on M defined at points of D. Suppose X has a C^1 flow in the following sense: there is an action $F:\boldsymbol{R}\times M\to M$ satisfying the conditions of Theorem 2, $F_t: D\to D$ and*

$$d/dt\, F_t(m)\big|_{t=0} = X(m) \quad \text{for } m\in D.$$

Then integral curves of X are unique: for any differentiable $c(t)$ (not a priori C^1) with $c'(t)=X(c(t))$ we have $c(t)=F_t(c(0))$.

Many interesting nonlinear semigroups F_t are actually smooth for t fixed (but of course, as in the linear case, need not be smooth jointly), for example smooth perturbations of linear generators, and the flows defined by the Euler and Navier-Stokes equations (on suitable spaces; [5]).

The idea of Theorem 3 is to consider $H(t)=F_{-t}(c(t))$ and to show H is differentiable and $H'(t)=0$. This is done by writing

$$(1/h)[H(t+h) - H(t)] = (1/h)[F_{-t}F_{-h}c(t+h) - F_{-t}F_{-h}c(t)] + (1/h)\int_0^1 DF_{-(t+h)}(c(t)+sv)\cdot v\, ds,$$

where $v=c(t+h)-c(t)$, and using joint continuity of $DF_t(x)\,u$ (by Theorem 2).

3. **Generalization to semigroups.** In generalizing to semigroups of

maps F_t, $t>0$, we have the following result.

THEOREM 4. *Let M be a metric space and F_t, $t>0$, a semigroup of continuous mappings of M to M such that for all $x\in M$, $t\mapsto F_t(x)$ is continuous. Then $F_t(x)$ is jointly continuous. The same result is true if M is a (paracompact) finite dimensional manifold, F_t is for each t, continuous and proper and $t\mapsto F_t(x)$ is (Borel) measurable.*

Analogues of Theorems 2 and 3 remain valid for such semi-group actions.

The first and last parts are proved much like the group case. The measurable case (which would also work for locally compact group actions) goes as follows: let $\boldsymbol{E}$ be the space of continuous functions on M which vanish at ∞ and set $U_t:\boldsymbol{E}\to\boldsymbol{E}$, $f\mapsto f\circ F_t$. Then U_t is a linear semigroup. It is not hard to see that U_t is strongly measurable and as $\boldsymbol{E}$ is separable, U_t is strongly continuous [8]. Using normality of M we can deduce the result for F_t.

One cannot deduce joint continuity at $t=0$. There is a counter-example for $M=\boldsymbol{R}^2$.

4. **Joint smoothness of actions.** An action π such that π_g is smooth will not in general be jointly smooth. (Any group generated by an unbounded linear operator is an example.) But for actions on *finite* dimensional manifolds this is valid. This is the basic result of Bochner and Montgomery [1]. However our proof yields more general results (as we shall see).

THEOREM 5. *Let G be a finite dimensional Lie group acting on a finite dimensional manifold M. Suppose π is separately (hence jointly) continuous and π_g is of class C^k for each $g\in G$ (here $1\leqq k\leqq\infty$). Then $\pi:G\times M\to M$ is jointly of class C^k.*

In particular a flow F_t on M which is, for each t, of class C^k is generated by a C^{k-1} vector field on M.

In general the generator need not be C^k. For example on $\boldsymbol{R}$ the vector field $X(x)=1$ if $x\leqq 1$, $X(x)=x$ if $x\geqq 1$ generates a flow F_t which is, for each fixed t, a C^1 diffeomorphism, but X is not C^1, only Lipschitz.

PROOF OF THEOREM 5. Let X be the C^k functions on M with the C^k topology. Then X is a sequentially complete locally convex space. Define $U_g:X\to X$ by $U_g(f)(x)=f(g^{-1}x)$. By hypothesis, U_g maps X to X. By joint continuity of π, $T\pi$, $\cdots$, $T^k\pi$ (Theorem 2), $\{U_g\}$ is locally equicontinuous. Thus (by arguments standard in semigroup theory; [8]) there is a dense set D^∞ in X such that if $f\in D^\infty$, the map

$g \mapsto U_g f$ from G to X is C^∞. By denseness of D^∞, there exists $f_1, \cdots, f_n \in D^\infty$ which form a local chart at $x_0 \in X$ (use the Jacobian). It follows easily that $g \mapsto g^{-1}x_0$ is C^k for g near e and by the group property for all g. Thus we have proven π is separately C^k. But by considering the tangent actions we get that π is jointly C^k.

We mention that Theorem 5 implies that a continuous homomorphism of Lie groups is necessarily C^∞, a well-known result. (If $\alpha: G \rightarrow H$ is a homomorphism, apply Theorem 5 to the action $\alpha: G \times H \rightarrow H$, $\pi(g, h) = \alpha(g) \cdot h$.)

5. **Application to diffeomorphism groups.** We work with the H^s case. Similar results are true in the C^k and Hölder cases. Let M be a compact manifold (possibly with boundary) and $\mathfrak{D}^s$ the group of H^s diffeomorphisms, $s > n/2 + 2$, $n = \dim M$. It is known (see [5]) that $\mathfrak{D}^s$ is a Hilbert manifold and a topological group. Moreover, $\mathfrak{D}^s$ admits an exponential map. That is, an H^s vector field Y on M has a flow consisting of H^s diffeomorphisms and these form a C^1 curve in $\mathfrak{D}^s$. Let $\mathfrak{D} = \mathfrak{D}^\infty$ be the group of C^∞ diffeomorphisms. Here we prove:

THEOREM 6. *A continuous one parameter subgroup of $\mathfrak{D}^s$ is a C^1 curve in $\mathfrak{D}^{s-1}$. A continuous one parameter subgroup of $\mathfrak{D}$ is a C^∞ curve.*

Actually the subgroup need only be measurable (in the sense given below).

Thus $\mathfrak{D}$ is like a Lie group (we remarked on Theorem 6 for Lie groups above) but it is infinite dimensional, locally Fréchet.

The proof of Theorem 6 is as follows: let X be the H^s functions on M and define $U_t: X \rightarrow X$ by $U_t(f) = f \circ F_t$. Standard facts about composition of H^s maps with H^s diffeomorphisms shows U_t maps X to X and is continuous (measurability of U_t of course suffices to guarantee continuity). See [5]. As in Theorem 5 we get a local H^s chart $f_1, \cdots, f_n$ such that $t \mapsto f_i \circ F_t$ is differentiable in H^s; (it is known (after Ebin) that the Jacobian test is valid to determine local H^s diffeomorphisms). This implies that the generator Y is H^{s-1} (again this cannot be improved to H^s). The statement for $\mathfrak{D}$ then follows from that for $\mathfrak{D}^s$ by standard arguments in global analysis; cf. [5, §2].

6. **A nonlinear generalization of a theorem of M. H. Stone.** Half of Stone's theorem asserts that the generator of a continuous one parameter unitary group is selfadjoint [8, p. 253]. We give a theorem on nonlinear semigroups which includes this as a special case.

The context of the results is in nonlinear Hamiltonian systems. We assume our flows are globally defined and the manifolds are flat for simplicity.

The proofs make use of the continuity and smoothness results above and further analysis (details will be given in [3]).

Let E be a Banach space and $F_t: E \to E$ a one parameter group of (nonlinear) mappings and assume for each fixed t, F_t is of class C^2, and is continuous in t (cf. Theorem 2). Let $A: D_A \subset E \to E$ be the generator of F_t; D_A consists of those $x \in E$ such that $d/dt\ F_t(x)\big|_{t=0} = Ax$ exists. Let B be the generator of the tangent flow TF_t. It is easy to see that $B(x, h) = (A_x, B_x \cdot h)$ for a linear operator B_x with domain denoted D_x. (One should think of B_x as the derivative of A at x.) As a preliminary result, we have

THEOREM 7. *Under the above hypotheses, we have* (DF = *derivative of* F):

(i) $x \in D_A$ *implies* $F_t(x) \in D_A$ *and* $A(F_t(x)) = DF_t(x) \cdot A(x)$.

(ii) D_x *is dense in* E *for each* $x \in D_A$.

(iii) *The operators* A *and* B_x *are closed, i.e. the graphs* Γ_A *and* Γ_{B_x} *in* $E \times E$ *are closed.*

Now assume E has a weak symplectic form ω, i.e. a skew symmetric, continuous bilinear form such that $\omega(h, k) = 0$ for all k implies $h = 0$, and F_t is symplectic, i.e.

$$\omega(DF_t(x)h,\ DF_t(x) \cdot k) = \omega(h, k).$$

We need, in addition, the technical assumption that there are local symplectic charts on E which linearize the domain D_A (this is related to Γ_A being a submanifold of $E \times E$).

THEOREM 8. *Under these assumptions, for each* $x \in D_A$, B_x *is a skew adjoint operator* (*with respect to* ω).

Of course skew adjoint is in the same strong sense as selfadjoint.

This includes Stone's theorem by taking $\omega(h, k)$ to be the imaginary part of the inner product $\langle h, k \rangle$ on the given Hilbert space.

NOTE ADDED IN PROOF. We wish to point out the following improvement of Theorem 4. Let M be a separable metric space and let F_t, $t > 0$, be a semigroup of continuous mappings of M to M such that, for all $x \in M$, the map $t \mapsto F_t x$ is Borel measurable. Then $F_t(x)$ is jointly continuous. This can be proved by adapting an idea which seems to go back to Banach. Fix $x \in M$; then the Borel mapping $t \mapsto F_t(x)$ is continuous when restricted to the complement of some first-category set C in $(0, \infty)$ [2]. Suppose that $t_0 > 0$ and $t_n \to t_0$. We claim that $F_{t_n}(x) \to F_{t_0}(x)$. We can assume that, for all n, $t_n > \frac{1}{2} t_0$. We can then find s with $0<s<\frac{1}{2}t_0$ such that $t_n-\frac{1}{2}t_0+s\notin C$ for $n=0, 1, 2 \cdots$. Therefore $F_{t_n-t_0/2+s}(x)\rightarrow F_{t_0/2+s}(x)$. By applying the continuous map $F_{t_0/2-s}$ to this limit, we deduce the asserted continuity of $F_t(x)$ at t_0. Thus we have separate continuity, and joint continuity now follows from Theorem 4.

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