NEW SPHERE PACKINGS IN DIMENSIONS 9–15

BY JOHN LEECH AND N. J. A. SLOANE

Communicated by H. S. M. Coxeter, March 13, 1970

New sphere packings in Euclidean spaces E^9 to E^{15} are constructed using nonlinear error-correcting codes and Steiner systems. The new packings and the previously best packings are collected in Table 1, which thus supersedes part of the table given by Leech [1].

1. **Sphere packings from distance 4 codes.** Let an (n, M, d) error-correcting code be a set of M binary vectors of length n such that any two vectors differ in at least d places. Given such a code with $d=4$ a packing of unit spheres in E^n may be obtained by the following

Construction. $\mathbf{x}=(x_1, \cdots, x_n)$ is a center of the packing if and only if $\mathbf{x}$ is componentwise congruent modulo 2 to a vector in the code.

Let the center density of a packing denote the fraction of space covered by the spheres divided by the content of a unit sphere. It is easily seen that this packing has center density $M2^{-n}$ and that the number of spheres touching the sphere with center congruent to a codeword $\mathbf{c}$ is equal to $2n+16A(\mathbf{c})$, where $A(\mathbf{c})$ is the number of codewords differing from $\mathbf{c}$ in exactly four places. The codes used here are nonlinear codes and so the packings obtained are nonlattice packings.

Nonlinear single error-correcting (8, 20, 3), (9, 38, 3), (10, 72, 3) and (11, 144, 3) codes have been given by Golay [2] and Julin [3]. By annexing a 0 at the end of codewords of even weight and a 1 at the end of codewords of odd weight they are made into codes with $d=4$. The packings $P9a$, $P10a$, $P11a$, $P12a$ of Table 1 are then obtained by applying the above construction to these codes. The codes are not unique and several inequivalent versions of the packings are possible.

It is worth remarking that these codes have more codewords (i.e., are more densely packed!) than any group code of the same length and minimum distance. They have a simple structure, and have recently been generalized to give single error-correcting codes of all lengths $2^m \leqq n < 3\cdot 2^{m-1}$, $m \geqq 3$, having more codewords than any comparable group code [4], [5].

AMS 1969 *subject classifications.* Primary, 5245.

Key words and phrases. Sphere-packing, packing, codes.

2. **Packings in E^{13}.** The technique of stacking layers packed in E^{n-1} so as to obtain dense packings in E^n is well known when $n=3$ and has been used in [6], [7] to obtain dense nonlattice packings in E^5, E^6, E^7. The new lattice packing K_{13} in E^{13} is obtained by applying this technique to K_{12}, and similarly the nonlattice packing $P13a$ is obtained by stacking layers of $P12a$.

Let Λ_{24} denote Leech's dense lattice packing in E^{24} and Λ_n the main sequence of sections of Λ_{24}, each of which is a densest section of the one above. The packings K_{11}–K_{13} are part of a branch sequence of sections $K_{18}=\Lambda_{18}$, K_{17}, $\cdots$, K_7, $K_6=\Lambda_6$. See [4] for details.

3. **Sphere packings from Steiner systems.** The Steiner system $S(3, 4, 10)$ [8] gives 30 codewords of length 10, weight 4 and Hamming distance at least 4; this may be extended to a (10, 36, 4) code by including the zero codeword and five codewords of weight 8. The packing $P10b$ in E^{10} is obtained by applying the construction to this code.

In E^{11} a local arrangement $P11b$ of 576 spheres touching one sphere is found by taking the 500 centers of $P10b$ with the eleventh coordinate zero, and also centers with the eleventh coordinate $\pm\frac{1}{2}\sqrt{6}$ and the first ten coordinates $\pm\frac{1}{2}$ with the signs forming the (10, 38, 4) code.

THEOREM. *Packings $P9a$, $P10a$, $P10b$ and $P11a$ are sections of $P12a$.*

The Steiner system $S(3, 4, 14)$ [9] gives 91 words of length 14, weight 4 and Hamming distance at least 4 apart and hence a local arrangement $P14a$ of 1484 spheres touching one sphere. This can be improved by taking the best section of $P14a$ in E^{13} in which 1066 spheres touch one sphere, and then packing adjacent layers based on the shortened Hamming code (13, 256, 4). In this way the local arrangement $P14b$ is obtained in which 1582 spheres touch one sphere.

The local arrangement $P15a$ is obtained in a similar way from $P14a$.

4. **Remarks.** 1. New packings in dimensions $n=2^m$, $m\geqq 6$, have also been obtained, with density Δ satisfying $\log_2\Delta\sim-\frac{1}{2}n\log_2\log_2 n$. Further details about all these packings will be given in [4].

2. The packings $P10a$, $P11a$, and $P13a$ seem to be the first examples of nonlattice packings with density greater than that of the densest known lattice packings. However, it is still an open question whether these are the densest possible lattice packings in these dimensions.

3. In the next-to-last column of the table, B indicates that both a lattice and a nonlattice packing with these parameters are known. L

TABLE 1. *Sphere packings in dimensions 9–15*

Space E^n	*Name of packing*	*Center density (truncated)*	*Number of spheres touched*			*Reference*	*Lattice, nonlattice, or both*	*Remarks*
			Maximum	*Average*	*Upper bound* [1]			
E^9	$\Lambda_9 = T_9$	$2^{-4\frac{1}{2}} = .04419$	272	272	401	[11]	*B*	
	P9a	$2^{-7} \cdot 5 = .03906$	306	$235\frac{3}{5}$			*N*	From (9, 20, 4) code
E^{10}	$\Lambda_{10} = \Phi_{10}$	$2^{-4} \cdot 3^{-\frac{1}{2}} = .03608$	336	336	648	[12]	*B*	
	P10a	$2^{-9} \cdot 19 = .03710$	372	$353\frac{9}{19}$			*N*	From (10, 38, 4) code
	P10b	$2^{-8} \cdot 3^2 = .03515$	500	$340\frac{1}{3}$			*N*	From *S* (3, 4, 10)
E^{11}	$\Lambda_{11} = J_{11}$	$2^{-5} = .03125$	438	438	1035	[1]	*B*	
	K_{11}	$2^{-1} \cdot 3^{-2\frac{1}{2}} = .03207$	432	432		[13]	*L*	
	P11a	$2^{-8} \cdot 3^2 = .03515$	566	$519\frac{7}{9}$			*N*	From (11, 72, 4) code
	P11b		576					A local arrangement only

TABLE 1 (*cont.*)

Space E^n	*Name of packing*	*Center density (truncated)*	*Number of spheres touched*			*Reference*	*Lattice nonlattice, or both*	*Remarks*
			Maximum	*Average*	*Upper bound* [1]			
E^{12}	$\Lambda_{12}=J_{12}$	$2^{-5}=.03125$	648	648	1637	[14]	*B*	
	K_{12}	$3^{-3}=.03703$	756	756		[14]	*L*	
	*P*12*a*	$2^{-8}\cdot 3^2=.03515$	840	$770\frac{2}{3}$			*N*	From (12, 144, 4)
E^{13}	Λ_{13}	$2^{-5}=.03125$	906	906	2569	[10]	*B*	
	K_{13}	$2^{-1}\cdot 3^{-2\frac{1}{2}}=.03207$	918	918			*B*	From K_{12}
	*P*13*a*	$2^{-8}\cdot 3^2=.03515$	1130	$1060\frac{2}{3}$			*N*	From *P*12*a*
E^{14}	Λ_{14}	$2^{-4}\cdot 3^{-\frac{1}{2}}=.03608$	1422	1422	4003	[10]	*B*	
	*P*14*a*		1484					A local arrangement only
	*P*14*b*		1582					A local arrangement only
E^{15}	Λ_{15}	$2^{-4\frac{1}{2}}=.04419$	2340	2340	6198	[10]	*B*	
	*P*15*a*		2564					A local arrangement only

indicates that at present only a lattice packing is known and N that only a nonlattice packing is known.

ACKNOWLEDGMENT. We are grateful to D. S. Whitehead for many helpful conversations.

REFERENCES

1. J. Leech, *Notes on sphere packings*, Canad. J. Math. **19** (1967), 251–267. MR **35** #878.

2. M. J. E. Golay, *Binary coding*, Trans. IRE **PGIT-4** (1954), 23–28. MR **19**, 622.

3. D. Julin, *Two improved block codes*, IEEE Trans. Information Theory **IT-11** (1965), 459.

4. J. Leech and N. J. A. Sloane, *Sphere packing and error-correcting codes* (to appear).

5. N. J. A. Sloane and D. S. Whitehead, *A new family of single error-correcting codes*, IEEE Trans. Information Theory (to appear).

6. J. Leech, *Five-dimensional non-lattice sphere packings*, Canad. Math. Bull. **10** (1967), 387–393. MR **36** #3236.

7. ———, *Six and seven dimensional non-lattice sphere packings*, Canad. Math. Bull. **12** (1969), 151–155.

8. E. Witt, *Über Steinersche Systeme*, Abh. Math. Sem. Hansischen Univ. **12** (1938), 256–264.

9. Haim Hanani, *On quadruple systems*, Canad. J. Math. **12** (1960), 145–157. MR **22** #2558.

10. J. Leech, *Some sphere packings in higher space*, Cand. J. Math. **16** (1964), 657–682. MR **29** #5166.

11. A. Korkine and G. Zolotareff, *Sur les formes quadratiques*, Math. Ann. 6 (1873), 366–389.

12. T. W. Chaundy, *The arithmetic minima of positive quadratic forms*. I, Quart. J. Math. Oxford Ser. **17** (1946), 166–192. MR **8**, 137.

13. E. S. Barnes, *The construction of perfect and extreme forms*. II, Acta. Arith. **5** (1959), 205–222. MR **21** #6360.

14. H. S. M. Coxeter and J. A. Todd, *An extreme duodenary form*, Canad. J. Math. **5** (1953), 384–392. MR **14**, 1066.

UNIVERSITY OF STIRLING, STIRLING, SCOTLAND

BELL TELEPHONE LABORATORIES, INC., MURRAY HILL, NEW JERSEY 07974