# ON THE FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS

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Let $X$ be a locally convex topological vector space, $Y$ a real Banach space, $f$ a mapping (in general, nonlinear) of $X$ into $Y$. In several recent papers ([5], [6], [7]), Pohožaev has studied the concept of *normal solvability* or the *Fredholm alternative* for mappings $f$ of class $C^1$. If $A_x = f_x'$ is the continuous linear mapping of $X$ into $Y$ which is the derivative of $f$ at the point $x$ of $X$, $A_x^*$ the adjoint mapping of $Y^*$ into $X^*$, his principal results assert that if the nullspace $N(A_x^*)$ is trivial for every $x$ in $X$, and if one of the two following hypotheses hold:

(1) $Y$ is reflexive and $f(X)$ is weakly closed in $Y$;

(2) $Y$ is uniformly convex and $f(X)$ is closed in $Y$;

then the image $f(X)$ of $f$ must be all of $Y$.

It is our purpose in the present paper to considerably sharpen and generalize these results by use of a different and more transparent argument. In particular, we establish a corresponding theorem for an arbitrary Banach space $Y$ and $f(X)$ closed in $Y$, allow exceptional points $x$ in $X$ at which the hypothesis on $N(A_x^*)$ may not hold, and derive this theorem from a basic theorem on general rather than differentiable mappings. The techniques which we apply below may be extended to infinite-dimensional manifolds and may be localized to prove the openness of $f$ under stronger hypotheses (as we shall do in another more detailed paper).

To state our basic theorem, we use the following definition:

DEFINITION 1. *Let $X$ be a real vector space, $f$ a mapping of $X$ into the real Banach space $Y$, $x$ a point of $X$. Then the element $v$ of the unit sphere $S_1(Y)$ of $Y$ is said to lie in the set $R_x(f)$ of asymptotic directions for $f$ at $x$ if there exists $\xi \neq 0$ in $X$ and a sequence $\{\gamma_j\}$ of positive numbers with $\gamma_j \to 0$ as $j \to \infty$ such that for each $j$, $f(x+\gamma_j\xi) \neq f(x)$, while*

$$\|f(x+\gamma_j\xi) - f(x)\|^{-1}(f(x+\gamma_j\xi) - f(x)) \to v \qquad (j \to \infty).$$

Our basic general result is the following:

THEOREM 1. *Let $X$ be a real vector space, $Y$ a real Banach space, $f$ a*

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*mapping of $X$ into $Y$ such that $f(X)$ is closed in $Y$. Suppose that there exists a finite subset $N$ of $X$ such that for all $x$ in $X-N$, the set $R_x(f)$ of asymptotic directions of $f$ at $x$ is dense in the unit sphere of $Y$.*

*Then $f(X)=Y$.*

We note that the hypothesis of Theorem 1 does not assume that $f$ is continuous in any topology, not to speak of being differentiable. On the other hand, the following lemma indicates how we may derive hypotheses on the asymptotic set of directions at a given point from hypotheses on the derivative if it exists:

LEMMA 1. *Let $X$ be a locally convex topological vector space, $Y$ a Banach space, and let $f$ be a mapping of an open subset of $X$ into $Y$ which is differentiable in the Gateaux sense at the point $x$ with its derivative $A_x$ a continuous linear mapping of $X$ into $Y$. Suppose that the nullspace $N(A_x^*)=\{0\}$, where $A_x^*$ is the adjoint map of $A_x$ carrying $Y^*$ into $X^*$.*

*Then $R_x(f)$ is dense in the unit sphere of $Y$.*

PROOF OF LEMMA 1. If $N(A_x^*)=\{0\}$, then the range $R(A_x)$ of $A_x$ is dense in $Y$, and in particular in the unit sphere of $Y$. Let $y$ be a point of the dense set $R(A_x)\cap S_1(Y)$ in $S_1(Y)$. If $y=A_x(\xi)$ for $\xi$ in $X$, then $\epsilon^{-1}[f(x+\epsilon\xi)-f(x)]\to y$ as $\epsilon\to 0$. It follows obviously since $\|y\|=1$ that

$$\|f(x+\epsilon\xi)-f(x)\|^{-1}[f(x+\epsilon\xi)-f(x)]\to y \qquad (\epsilon\to 0).$$

Hence $y$ lies in $R_x(f)$. q.e.d.

THEOREM 2. *Let $X$ be a real locally convex topological vector space, $f$ a mapping of $X$ into the real Banach space $Y$. Let $N$ be a finite subset of $X$, and suppose that $f$ is differentiable in the Gateaux sense on $X-N$ and that if $A_x$ is the derivative of $f$ at the point $x$, $A_x$ is a continuous linear mapping of $X$ into $Y$ and $N(A_x^*)=\{0\}$ for each $x$ in $X-N$. Suppose further that $f(X)$ is closed in $Y$.*

*Then $f(X)=Y$.*

Theorem 2 is an immediate consequence of Theorem 1 and Lemma 1. We note that for $X$ a Banach space, mappings $f$ which are continuously Fréchet differentiable with the range of $A_x$, the derivative of $f$, closed for each $x$ in $X$ $(N=\varnothing)$ and with $N(A_x^*)=\{0\}$ for all $x$ in $X$, the result of Theorem 2 is a simple consequence of a generalization of the implicit function theorem given by Graves [4]. In particular, if $f$ is a nonlinear Fredholm mapping in the sense of Smale [8], $A_x$ will have closed range for each $x$. However, the need for a different argument in general is forced by the fact that we do not assume $R(A_x)$ closed in $Y$, nor continuous differentiability, nor that $N=\varnothing$.

1. We base the proof of Theorem 1 on the following result in the geometry of Banach spaces:

THEOREM 3. *Let $Y$ be a Banach space, $S$ a closed subset of $Y$ with $S = Y$. Let $s_0$ be a point in the boundary of $S$ in $Y$, and let $\epsilon > 0$ be given. Then there exists a point $s_1$ in the boundary of $S$ in $Y$ with $\|s_0 - s_1\| < \epsilon$, an element $v_0$ of $S_1(Y)$, and $\zeta > 0$ such that*

$$C = \{y \mid y \in Y, 0 < \|y - s_1\| < \zeta, \| [\|y - s_1\|^{-1}(y - s_1) - v_0]\| < \zeta\}$$

*does not intersect $S$.*

In intuitive terms, Theorem 3 states that at some point $s_1$ of $S$ near $s_0$, there exists a cone with interior with vertex at $s_1$ which intersects $S$ only at $s_1$ in the neighborhood of $s_1$.

PROOF OF THEOREM 1 FROM THEOREM 3. We suppose that $S = f(X) \neq Y$. Let $S_1 = \text{bdry}(S)$. Since $R_x(f)$ is dense in $Y$ for each point $x$ in $X - N$, it follows that $S$ is infinite. As a result, $S_1 - f(N)$ is nonempty. Indeed, suppose $S_1 = f(N)$ so that $S_1$ is finite, and let $y_0$ be a point in $S - f(N)$. For each ray $R$ which emanates from $y_0$, $R$ contains points of $S$ (namely $y_0$ itself) and can contain points of $Y - S$ only if $R$ intersects $\text{bdry}(S) = S_1$. If $S_1$ is finite, only a finite number of such rays can intersect $Y - S$. Hence, all the other rays are contained in $S$, $S$ is dense in $Y$, and since $S$ is closed in $Y$ by hypothesis, it follows that $S = Y$ contrary to our assumption.

We choose a point $s_0$ in $S_1 - f(N)$ and $\epsilon > 0$ with $\epsilon < \text{dist}(s_0, f(N))$. By Theorem 3, we can find a point $s_1$ in $\text{bdry}(S)$ and a suitable cone with vertex at $s_1$ intersecting $S$ only in $s_1$ in some neighborhood of $s_1$. Since $s_1$ lies in $S - f(N)$, there exists $x$ in $X - N$ such that $f(x) = s_1$. In particular, if $v_0$ and $\zeta$ are the data for the cone at $s_1$, we can find $v$ in $R_x(f)$ such that $\|v - v_0\| < \zeta/2$. Since $v$ lies in $R_x(f)$, there exists an element $\xi$ in $X$ and a sequence $\epsilon_j$ such that $\epsilon_j \to 0$, $f(x + \epsilon_j \xi) - f(x) \neq 0$, $f(x + \epsilon_j \xi) \to f(x)$, and

$$\|f(x + \epsilon_j \xi) - s_1\|^{-1}[f(x + \epsilon_j \xi) - s_1] \to v.$$

Set $y_j = f(x + \epsilon_j \xi)$ for each $j$. Then $y_j$ lies in $S$, is distinct from $s_1$ for each $j$, $y_j \to s_1$ as $j \to \infty$. Hence for $j$ sufficiently large, $\|y_j - s_1\| < \zeta$, while

$$\| \|y_j - s_1\|^{-1}(y_j - s_1) - v_0\| \leqq \|v - v_0\| + \zeta/2 < \zeta,$$

which contradicts the characteristic property of the cone given by Theorem 3. q.e.d.

2. We now give the proof of Theorem 3, thereby completing the argument as well for Theorems 1 and 2. This proof is based upon a device applied by Bishop and Phelps [1] to prove the density of sup-

port points for a bounded closed convex subset $C$ of $Y$. We note that for the case in which $S$ is convex, Theorem 3 is equivalent to the Bishop-Phelps result. The basic tool of the argument is contained in the following lemma:

LEMMA 2. *Let $Y$ be a Banach space, $S_0$ a closed bounded subset of $Y$. Suppose that $C_0$ is a closed cone with vertex at the origin in $Y$ such that for a given $y^*$ in $S_1(Y^*)$ and a constant $M>0$,*

$$\|y\| \leqq M(y^*, y) \tag{1}$$

*for all $y$ in $S_0$. (We use the notation $(w, y)$ for the pairing between an element $w$ of $Y^*$ and an element $y$ of $Y$.)*

*Then there exists an element $y_0$ of $S_0$ such that $S_0 \cap (y_0+C_0) = \{y_0\}$.*

PROOF OF LEMMA 2. We introduce a parting ordering on $S_0$ by letting $y \leqq y_1$ whenever $y_1 \in y+C_0$. The point $y_0$ of $S_0$ satisfies the conclusion of Lemma 2 if and only if it is maximal in this ordering. To prove the existence of a maximal element, it suffices by Zorn's Lemma to prove that every totally ordered subfamily $\{y_\alpha\}$ of $S_0$ has an upper bound in $S_0$. Consider any finite subfamily $\{y_1, y_2, \cdots, y_n\}$ of the subfamily. We may assume that $y_1 \leqq y_2 \leqq \cdots \leqq y_n$, i.e. $y_{j+1} \in y_j + C_0$ for each $j$. By the inequality (1), we have $\|y_{j+1}-y_j\| \leqq M(y^*, y_{j+1}-y_j)$.

If $k = \sup_{y \in S_0}\|y\|$, it follows that

$$\sum_{j=1}^{n-1} \|y_{j+1} - y_j\| \leqq M(y^*, y_n - y_1) \leqq 2Mk.$$

Since this inequality holds for each finite subfamily, it follows that the family $\{y_\alpha\}$ is countable and may be written as a sequence $\{y_j\}$ with $y_j \leqq y_{j+1}$ for each $j$. Then $\sum_{j=1}^{\infty} \|y_{j+1}-y_j\| \leqq 2Mk$, and $y_j \rightarrow y$ in $Y$, $y \geqq y_j$ for all $j$. Since $S_0$ is closed, $y$ lies in $S_0$ and is the desired upper bound for $\{y_\alpha\}$. q.e.d.

PROOF OF THEOREM 3. Let $s_0$ be a given point of bdry$(S)$, $\epsilon>0$. Since $s_0$ lies in bdry$(S)$, there exists a point $y_0$ of $Y-S$ such that $\|y_0-s_0\| < \epsilon/4$. For a given $y_0$, we then find a point $s_2$ of $S$ such that $\|s_2-y_0\| \leqq (4/3)\text{dist}(y_0, S)$. Thus if $d = \|y_0-s_2\|$, it follows that $d<\epsilon/3$.

Let $B$ be the closed ball of radius $d/4$ about $y_0$, and let

$$C_1 = \{y \mid y = (1-\lambda)s_2 + \lambda z, z \in B, \lambda \in [0, 1]\},$$
$$C_2 = \{y \mid y = \xi(z - s_2), z \in B, \xi \geqq 0\}.$$

Then $C_1$ is a closed, bounded, convex subset of $Y$ and $C_2$ is a closed cone in $Y$. Let $S_0 = S \cap C_1$. $S_0$ is a closed bounded subset of $Y$. Let $s \in S_0$. Then $s = (1-\lambda)s_2 + \lambda z_1$ for some $z_1$ in $B$, $\lambda$ in $[0, 1]$, and we have

$$3d/4 \leqq \operatorname{dist}(y_0, S) \leqq \|y_0 - s\| \leqq (1-\lambda)\|y_0 - s_2\| + \lambda\|z_1 - y_0\|$$
$$\leqq (1-\lambda)d + \lambda\, d/4,$$

so that $\lambda \leqq 1/3$. For any point $s$ in $S_0$ and any element $y$ of $C_2$ with $y = \xi(z - s_2)$, for $z$ in $B$ and $0 \leqq \xi \leqq 2/3$,

$$s + y = (1-\lambda)s_2 + \lambda z_1 + \xi(z - s_2)$$
$$= (1 - \lambda - \xi)s_2 + (\lambda + \xi)[\lambda(\lambda+\xi)^{-1}z_1 + \xi(\lambda+\xi)^{-1}z] \in C_1$$

since $\lambda + \xi \leqq 1$, $[\lambda(\lambda+\xi)^{-1}z_1 + \xi(\lambda+\xi)^{-1}z] \in B$ for $z$, $z_1$ in $B$. Moreover, for any element of the form $s+y$, $y = \xi(z - s_2)$ with $z$ in $B$, $\xi > 2/3$, we have

$$\|s - (s+y)\| = \|y\| > (2/3)\|z - s_2\| \geqq (2/3)(3d/4) = d/2.$$

We choose an element $y^*$ of $Y^*$ with $\|y^*\| = 1$ such that $(y^*, y_0 - s_2) > 0$ and then a constant $M > 0$ such that $M(y^*, y_0 - s_2) > d$. For this constant $M$, we form the closed cone $C_3$ given by $C_3 = \{y \mid \|y\| \leqq M(y^*, y)\}$. The two cones $C_2$ and $C_3$ have $(y_0 - s_2)$ in their interior. Hence, so does their intersection $C_0 = C_2 \cap C_3$.

We apply Lemma 2 to the closed bounded subset $S_0$ and the cone $C_0$ (which satisfies the inequality (1) since $C_0 \subset C_3$). We obtain a point $s_1$ of $S_0$ such that $S_0 \cap (s_1 + C_0) = \{s_1\}$. Consider any point $s$ of $S \cap (s_1 + C_0)$ with $\|s - s_1\| \leqq d/2$. Then $s$ lies in $s_1 + C_2$ and by our preceding remarks, $s$ lies in $C_1$. Hence $s$ lies in $S_0 = S \cap C_1$ and therefore $s = s_1$. Since $C_0$ is a cone with interior, we can replace it by a smaller cone of the form

$$C' = \{y \mid \|[\|y\|^{-1}y - v_0]\| < \zeta\} \cup \{0\}$$

with $\zeta < d/2$. Moreover,

$$\|s_1 - s_0\| \leqq \|y_0 - s_0\| + \|s_1 - y_0\| \leqq \epsilon/4 + d \leqq \epsilon/4 + \epsilon/3 < \epsilon. \quad \text{q.e.d.}$$

REMARKS. (1) The condition that $N$ be finite can obviously be replaced by the weaker condition that $f(N)$ should not be dense in $\operatorname{bdry}(f(X))$.

(2) The argument of Pohožaev in [7] is based upon the result of Edelstein [3] that in a uniformly convex set, the set of $y$ for a given closed subset $S$ such that $\operatorname{dist}(y, S)$ is assumed forms a set dense in $Y$.

(3) For a discussion of the general approach using local arguments like the theorem of Graves [4], see §2 of the writer's paper [2].

(4) The term "Fredholm alternative" is obviously not unambiguous in its potential uses for nonlinear operators. In particular, the present considerations have no point of contact with recent results of

Neças concerning the solvability of homogeneous operators of the form $T+\lambda S$ with $T$ strongly monotone and $S$ completely continuous.

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