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GROUP INVARIANCE IN NONLINEAR FUNCTIONAL ANALYSIS

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Introduction. If X is an infinite dimensional Banach space (or more generally, an infinite dimensional manifold), T and S two mappings of X into another space Y, the typical problems of nonlinear functional analysis ask about the set of points u in X for which $T(u) = S(u)$, or for which $T(u) = \lambda S(u)$ for some real λ, or for which $T(u) = y_0$ for a given y_0 in Y. Aside from basic structural hypotheses on the classes of mappings T and S considered (i.e. hypotheses that one operator or the other is compact, monotone, accretive, nonexpansive, proper, Fredholm, or whatever), in order to obtain nontrivial existence results for the desired solutions u, one must impose additional hypotheses in the large usually in the form of boundary conditions or asymptotic conditions (coerciveness, boundedness of inverse mappings, etc.). There is an alternative type of additional hypothesis, however, under which one obtains nontrivial results with the boundary or asymptotic conditions weakened or eliminated, namely the hypothesis that the nonlinear problem is invariant under a group G of transformations acting on the spaces X and Y with G having elements of finite order.

In another paper (Browder [5]), we have obtained results on the application of the Lusternik-Schnirelman theory to obtain infinitely many distinct solutions of the nonlinear eigenvalue problem $g'(u) = \lambda h'(u)$, where g' and h' are the Fréchet derivatives of real-valued functions g and h on an infinite dimensional Banach space B and the usual hypothesis that g and h are even functions is replaced by the

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hypothesis that for a given transformation group G of B with at least one element of finite order and with G having only 0 as a fixed point for any nontrivial element, we have $g(\phi(x))=g(x)$, $h(\phi(x))=h(x)$ for any x in $B-\{0\}$, $\phi\in G$.

It is our purpose in the present paper to give an extension within this framework of group invariance of the classical Borsuk-Ulam theorem which asserts that for an odd mapping of a sphere S^m about 0 into itself, the degree of the mapping is odd and hence different from zero. (For references to some recent infinite dimensional extensions and analytical applications of the Borsuk-Ulam theorem, we refer to the writer's recent note (Browder [8]).)

THEOREM 1. *Let T be a continuous mapping of $S_1(R^n)$ into itself for some $n\geqq 2$ (where $S_1(R^n)$ is the unit sphere about the origin in R^n). Suppose that there exists a group G of homeomorphisms of $S_1(R^n)$ on itself such that G has at least one nontrivial element of finite order while each nontrivial element of G acts without fixed points on $S_1(R^n)$. Suppose that for each ϕ in G, $T\phi=\phi T$.*

Then the degree of T is different from zero.

We derive Theorem 1 from the following slightly more precise result:

THEOREM 2. *Let T be a continuous map of $S_1(R^n)$ into itself. Suppose that there exists a homeomorphism ϕ of $S_1(R^n)$ onto itself such that $T\phi=\phi T$ while $\phi^p=$ the identity for a given prime $p\geqq 2$, and ϕ^j acts without fixed points on $S_1(R^n)$ for $1\leqq j\leqq p-1$.*

Then, if d is the degree of T, $d\equiv 1 \pmod p$, and $d\neq 0$.

The Borsuk-Ulam theorem corresponds to the case in which $\phi(x)=-x$ for every x in $S_1(R^n)$. Theorem 2 implies Theorem 1 since if T and G satisfy the hypotheses of Theorem 1 and if ψ is an element of order s of G, then for any nontrivial prime divisor p of s, $\phi=\psi^{s/p}$ satisfies the hypothesis of Theorem 2.

We may combine the result of Theorem 1 with various generalized theories of the topological degree for mappings of infinite dimensional Banach spaces (e.g. Browder [3], [4], [6], [8], Browder-Nussbaum [9], Browder-Petryshyn [10], [11], Nussbaum [16]), and we obtain thereby new existence theorems for a wide variety of nonlinear mappings in Banach spaces, as in the following:

THEOREM 3. *Let X be a Banach space, G a convex open subset of X which contains the origin, Γ the boundary of G in X. Let C be a compact mapping of $\mathrm{cl}(G)$ into X, (i.e. C maps $\mathrm{cl}(G)$ into a relatively compact* *subset of X and C is continuous). Suppose that there exists a bounded linear mapping R of X into X such that R maps Γ into* $\mathrm{cl}(G)$ *and for x in Γ, $R(C(x)) = C(R(x))$, and with the additional property that for some prime $p \geqq 2$, $R^p = I$ while for $1 \leqq j \leqq p-1$, R^j has only 0 as a fixed point. Suppose that $x \neq C(x)$, $x \in \Gamma$.*

Then $\deg(I-C, G, 0) \equiv 1 \pmod{p}$, *and there exists x_0 in G such that $C(x_0) = x_0$. Moreover, if $k_0 = \mathrm{dist}((I-C)(G), 0) > 0$, the closed ball of radius k_0 about the origin in X is contained in $(I-C)(\mathrm{cl}(G))$.*

Theorem 3 extends the original extension of the Borsuk-Ulam theorem to compact maps due to Krasnosel'skiĭ (cf. [14]) and may be carried over under appropriate hypotheses to various classes of noncompact mappings considered by the writer in [6] and elsewhere; semiaccretive, semicontractive, and in general, mappings defined by intertwined representations involving convex classes of maps obtained as the limits of nonsingular mappings. We shall give the detailed statement and discussion of these applications elsewhere, as well as the corresponding extension of the generalized degree theory based upon approximation schemes of Galerkin type for A-proper mappings as developed in Browder-Petryshyn [10], [11]. We content ourselves here with the statement of a result useful in applications to nonlinear elliptic problems:

THEOREM 4. *Let X be a reflexive separable Banach space, T a pseudomonotone mapping of X into X^* with X^* the conjugate space of X. Suppose that T is finitely continuous (i.e. continuous from each finite dimensional subspace of X to the weak topology of X^*) and that there exists a bounded linear mapping R of X into X having the property that $R^p = I$ for a given prime $p \geqq 2$, with R^j not having $(+1)$ as an eigenvalue for $1 \leqq j \leqq (p-1)$, such that $R^*T = TR$. (Here, $R^*: X^* \to X^*$ is the adjoint operator to R.)*

Let K_0 and k_0 be positive constants such that for $\|x\| = K_0$, $\|T(x)\| \geqq k_0$. Then for each w in X^* with $\|w\| \leqq k_0$, there exists x in X with $\|x\| \leqq K_0$ such that $T(x) = w$. In particular, if T^{-1} is a bounded mapping of X^* into X (i.e. maps bounded sets into bounded sets), then the range of T is all of X^*.

1. Since, as we have already observed, Theorem 2 implies Theorem 1, we proceed to the detailed discussion of Theorem 2. We assume therefore that for a given prime $p \geqq 2$, we have a periodic homeomorphism ϕ of S^n for a given n of period p such that for $1 \leqq j \leqq (p-1)$, ϕ^j acts without fixed points on S^n. Let $Y_{n,G}$ be the quotient space of S^n under the action of the cyclic group G of transformations of S^n generated by ϕ, π3, the quotient mapping. Then π is a covering mapping,

$Y_{n,G}$ is an n-dimensional manifold with S^n as its universal covering space if $n\geqq 2$. We note that if $p\neq 2$, the Lefschetz fixed point theorem together with the fixed point-free character of ϕ^j for $1\leqq j\leqq (p-1)$ implies that if d_0 is the degree of ϕ, then $1+(-1)^n d_0^j=0$ for all such j. Hence $d_0=(-1)^{n+1}$, $d_0=d_0^2$ which implies that n is odd and that $d_0=+1$. In particular, for p odd, ϕ is an orientation-preserving mapping and $Y_{n,G}$ is an orientable n-manifold. If $p=2$, $d_0=(-1)^{n+1}$ with no restriction on n, and ϕ is orientation-preserving if and only if n is odd.

We reduce the general case for $p=2$ to the orientation-preserving case by suspending the mapping f in the following fashion: Take S^n to be the unit sphere in R^{n+1} and consider the unit sphere S^{n+2} in R^{n+3} about the origin. Each element v in S^{n+2} may be written uniquely in the form $v=(ru, r_1w)$ with u in S^n, w in S^2 and $r^2+r_1^2=1$, $r\geqq 0$, $r_1\geqq 0$. Let $\psi(v)=(r\phi(u), -r_1w)$, $g(v)=(f(u), w)$. Then ψ is an orientation-preserving involution of S^{n+2}, g commutes with ψ, and the degrees of f and g coincide as mappings of S^n and S^{n+2}, respectively. Hence, we need merely prove the assertion of Theorem 2 for g and can assume without loss of generality that ϕ is orientation-preserving in every case and $n\geqq 2$.

We consider the singular homology and cohomology groups of S^n and $Y_{n,G}$ with coefficients in the integers Z, and in Z_p, the integers mod p. Since S^n and $Y_{n,G}$ are orientable n-manifolds, $H_n(S^n; Z)$ has a single generator α, and $H_n(Y_{n,G}; Z)$ has a single generator α_1, where these generators may be chosen with respect to concordant orientations so that if $\pi_{*n}:H_n(S^n; Z)\to H_n(Y_{n,G}; Z)$ is the homomorphism induced by the map π, then $\pi_{*n}(\alpha)=p\alpha_1$. If $f_{*n}:H_n(S^n; Z)\to H_n(S^n; Z)$ is the endomorphism induced by the map f, then $f_{*n}(\alpha)=d\alpha$, where d is the degree of the mapping f.

Because f commutes with ϕ, it induces a map h of $Y_{n,G}$ into $Y_{n,G}$ such that $h\pi=\pi b$. The map h of $Y_{n,G}$ induces an endomorphism of the fundamental group $\pi_1(Y_{n,G}, y_0)$ for a given base point y_0 in $Y_{n,G}$ as follows: For a given path C_0 from y_0 to $h(y_0)$ and for the homotopy class $[C]$ of a closed path C with initial and final point at y_0, $h_*([C])=[C_0^{-1}h(C)C_0]$. Since $\pi_1(Y_{n,G}, y_0)$ is isomorphic to the group G of covering transformations of the universal covering space S^n of $Y_{n,G}$, $\pi_1(Y_{n,G}, y_0)$ is abelian. Hence h_* is independent of the choice of the path C_0. Moreover, if we consider the natural homomorphism η of $\pi_1(Y_{n,G}, y_0)$ into $H_1(Y_{n,G}; Z)$, η is an isomorphism [**13**, Theorem 8.8.3, p. 348]. Since $f\phi=\phi f$, it follows by an elementary argument on the covering transformations that h_* is the identity endomorphism of $\pi_1(Y_{n,G}, y_0)$. Hence, the endomorphism $h_{*1}:H_1(Y_{n,G}; Z)\to H_1(Y_{n,G}; Z)$

is the identity endomorphism. There is a natural epimorphism of $H_1(Y_{n,G}; Z)$ onto $H_1(Y_{n,G}; Z_p)$ induced by the homomorphism γ of the coefficient groups which carries each integer z into z mod p. Hence, the endomorphism $h_{*1}:H_1(Y_{n,G}; Z_p)\rightarrow H_1(Y_{n,G}; Z_p)$ is the identity. Finally, by Pontrjagin duality, the cohomology endomorphism $h^{*1}:H^1(Y_{n,G}; Z_p)\rightarrow H^1(Y_{n,G}; Z_p)$ is the identity endomorphism. To proceed further, we apply the following well-known result on the cohomology ring of $Y_{n,G}$ with coefficients in Z_p:

LEMMA 1. *For $p>2$, $H^*(Y_{n,G}; Z_p)$ is the quotient of the tensor product of an exterior algebra on a one-dimensional element w_1 with a polynomial algebra on a two-dimensional element w_2 by the ideal of elements of dimension $>n$. Here, $w_2=-\beta(w_1)$, where β is the Bockstein homomorphism of $H^1(Y_{n,G}; Z_p)$ into $H^2(Y_{n,G}; Z_p)$ associated with the exact sequence, $0\rightarrow Z_p\rightarrow Z_{p^2}\rightarrow Z_p\rightarrow 0$.*

If $p=2$, $H^*(Y_{n,G}; Z_p)$ consists of all polynomials in a one-dimensional element w_1, module the ideal generated by w_1^{n+1}.

PROOF OF LEMMA 1. Lemma 1 is obtained from the consideration of the spectral sequence of the covering map π. For the case where X is an infinite-dimensional Banach space and $Y_{\infty,G}$ is the homogeneous space associated with the action of G on the unit sphere $S_1(X)$, $Y_{\infty,G}$ is an Eilenberg-MacLane space $K(G, 1)$ and the corresponding results without truncation at dimension n are given in [**19**, p. 68]. As has been pointed out to the writer by A. Liulevicius, if the action of G on S^n is differentiable, we can easily imbed $Y_{n,G}$ in an Eilenberg-MacLane space $K(G, 1)$ such that $K(G, 1)-Y_{n,G}$ is a relative CW complex with no cells of dimension $\leqq n$. It follows that the homomorphisms $H^k(K(G, 1); Z_p)\rightarrow H^k(Y_{n,G}; Z_p)$ are isomorphisms for $k<n$, and we get an epimorphism for $k=n$. In the latter case, both groups are isomorphic to Z_p and we get an isomorphism there also.

PROOF OF THEOREM 2 COMPLETED. Since $h^*:H^*(Y_{n,G}; Z_p)\rightarrow H^*(Y_{n,G}; Z_p)$ is a ring homomorphism, in order to prove that h^* is the identity homomorphism, it suffices to establish this on its ring generators. For w_1, $h^{*1}(w_1)=w_1$ as we have already noted. Since β commutes with h^*, $h^{*,2}(w_2)=w_2$. Thus by Lemma 1, $h^*(w)=w$ for every element w of $H^*(Y_{n,G}; Z_p)$.

If $h^{*,n}(\alpha_2)=k\alpha_2$ for a generator α_2 of $H^n(Y_{n,G}; Z)$, it follows that since the homomorphism induced by γ carries $H^n(Y_{n,G}, Z)$ onto $H^n(Y_{n,G}; Z_p)$, h^{*n} acts as multiplication by $\gamma(k)$ on $H^n(Y_{n,G}; Z_p)$. Hence, $\gamma(k)=1$, (i.e. $k=1$ (mod p)). On the other hand, $h_{*n}(\alpha_1)=k\alpha_1$.

Hence

$$h_{*n}(\pi_{*n}(\alpha)) = h_{*n}(p\alpha_1) = kp\alpha_1,$$

and

$$h_{*n}(\pi_{*n}(\alpha)) = \pi_{*n}(f_{*n}(\alpha)) = \pi_{*n}(d\alpha) = pd\alpha_1.$$

Thus, $k=d$, and $d\equiv 1 \pmod p$. q.e.d.

REMARK. The only use of Lemma 1 is to establish that h^* is the identity homomorphism on $H^*(Y_{n,G}; Z_p)$. For the case in which G acts differentiably, this can be obtained from the study of the Smith homomorphisms, as in the proof of Theorem (7.6) of Chapter 13 of [2]. The generalized lens spaces $Y_{n,G}$ were considered for special actions of G by De Rham [12] (see also [15, pp. 15–16, 180]).

PROOF OF THEOREM 3. Given $\epsilon>0$, we may construct a finite dimensional subspace F of X which is invariant under R and a retraction S of $\mathrm{cl}(C(\mathrm{cl}(G)))$ into F such that $\|S(x)-x\|<\epsilon$ for x in $\mathrm{cl}(C(\mathrm{cl}(G)))$ and $SR=RS$. For sufficiently small ϵ, it follows from the theory of the Leray-Schauder degree that $\deg(I-C, G, 0) =\deg(I-SC, G\cap F, 0)$. On the other hand, the finite dimensional mapping SC commutes with R since both S and C do. Applying a slight variant of Theorem 2, we find that $\deg(I-SC, G\cap F, 0)\equiv 1 \pmod p$. Hence, the result of Theorem 3 follows. q.e.d.

PROOF OF THEOREM 4. Since X is reflexive and separable, we may find an increasing sequence $\{F_k\}$ of finite dimensional subspaces of X such that their union is dense in X while each F_k is invariant under the action of the linear map R. Let J be the normalized duality mapping of X into X^* corresponding to a norm on X in which X is locally uniformly convex and X^* is strictly convex. For $\epsilon>0$, ϵ sufficiently small, $T+\epsilon J$ has no zeroes on the boundary of the ball of radius K_0 about the origin in X, while $T+\epsilon J$ is A-proper in the sense of Browder-Petryshyn [11] with respect to the injective approximation scheme defined by the sequence $\{F_k\}$. For each k, let j_k be the injection map of F_k into X, j_k^* the adjoint projection map of X^* onto F_k^*. Then $T_k=j_k^*Tj_k$ maps F_k into F_k^*, and if we let $R_k=R|_{F_k}$ and R_k^* the adpoint map of F_k^* into F_k^*, then $R_k^*T_k=T_kR_k$. If $T_{\epsilon,k}=j_k^*(T+\epsilon J)j_k$, then for ϵ sufficiently small,

$$\deg(T_k, B_{K_0}\cap F_k, 0) = \deg(T_{\epsilon,k}, B_{K_0}\cap F_k, 0).$$

By a slight variant of Theorem 2, $\deg(T_k, B_{K0}\cap F_k, 0)\equiv 1 \pmod p$. Hence, $\deg(T+\epsilon J, B_{k_0}, 0)\neq\{0\}$ for $\epsilon>0$, ϵ sufficiently small. q.e.d.

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