FREE BOUNDARY PROBLEMS FOR PARABOLIC EQUATIONS

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1. One dimensional problems. Denote by $D_i(T)$ $(1 \leqq i \leqq k)$ a 2-dimensional domain bounded by two curves $x = s_{i-1}(t)$, $x = s_i(t)$ where $0 < t < T$, and by the line segments $t = 0$, $b_{i-1} < x < b_i$ and $t = T$, $s_{i-1}(T) < x < s_i(T)$. Here $s_{i-1}(t) < s_i(t)$, $s_0(t) \equiv b_0$, $s_k(t) \equiv b_k$ where b_0, b_k are constants. Let

$$L_m u \equiv a^m(x, t) \frac{\partial^2 u}{\partial x^2} + b^m(x, t) \frac{\partial u}{\partial x} + c^m(x, t)u - \frac{\partial u}{\partial t} \qquad (m = 1, 2)$$

be parabolic operators with smooth coefficients and with $c^m(x, t) \leqq 0$. Suppose, for definiteness, that k is an even number. Consider the following problem: Find such curves $s_1, \cdots, s_{k-1}$ and functions u_1, u_2, that

$$(1.1) \qquad L_1 u_1 = f_1 \quad \text{in } D_1(T) \cup D_3(T) \cup \cdots \cup D_{k-1}(T),$$

$$(1.2) \qquad L_2 u_2 = f_2 \quad \text{in } D_2(T) \cup D_4(T) \cup \cdots \cup D_k(T),$$

$$(1.3) \qquad u_1(x, 0) = h_1(x) \quad \text{if } b_{i-1} < x < b_i, \qquad i = 1, 3, \cdots, k-1,$$

$$(1.4) \qquad u_2(x, 0) = h_2(x) \quad \text{if } b_{i-1} < x < b_i, \qquad i = 2, 4, \cdots, k,$$

$$(1.5) \qquad \text{either } u_1 = g_1 \quad \text{or} \quad \lambda_1 \frac{\partial u_1}{\partial x} + \mu_1 u_1 = g_1 \quad \text{for } x = b_0, \quad 0 < t < T,$$

$$(1.6) \qquad \text{either } u_2 = g_2 \quad \text{or} \quad \lambda_2 \frac{\partial u_2}{\partial x} + \mu_2 u_2 = g_2 \quad \text{for } x = b_k, \quad 0 < t < T,$$

$$(1.7) \qquad u_1 = u_2 = \Phi\left(\frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, s_i, \frac{ds_i}{dt}\right) \quad \text{on } x = s_i(t), \quad 0 < t < T \qquad (1 \leqq i \leqq k-1),$$

$$(1.8) \qquad \Psi\left(u_1, u_2, \frac{\partial u_1}{\partial x}, \frac{\partial u_2}{\partial x}, s_i, \frac{ds_i}{dt}\right) = 0 \quad \text{on } x = s_i(t), \quad 0 < t < T \qquad (1 \leqq i \leqq k-1);$$

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here f_i, h_i, λ_i, μ_i, g_i, Φ and Ψ are given functions. Such a problem is called a *k-phase free boundary problem*. The curves $x = s_i(t)$ $(1 \leqq i \leqq k-1)$ are called *free boundaries*.

One can also formulate the analogous problem in case k is odd, in case $s_0(t)$ and $s_k(t)$ are given nonconstant functions, and also in case $b_0 = -\infty$ or $b_k = \infty$. The problem can further be generalized by considering k parabolic operators L_m and replacing (1.1), (1.2) by

$$L_m u_m = f_m \quad \text{in } D_m(T) \qquad (1 \leqq m \leqq k).$$

One then makes the obvious changes in the other conditions (1.3)–(1.8). Finally, in some problems, Φ or Ψ are not pointwise functions of their variables but are rather functionals.

Some natural questions arise regarding the above formulated free boundary problem:

(i) Does there exist a unique solution for any T?

(ii) Does the solution depend monotonically upon the initial and boundary conditions?

(iii) Is the solution a continuous function of the data, in certain norms?

(iv) Does the solution have an asymptotic behaviour as $t \to \infty$?

(v) Often the parabolic equations or the boundary conditions at the free boundary depend on a parameter α. How does the solution depend on α?

The most studied free boundary problem is the *Stefan problem*. Here u_1 represents the temperature of water and u_2 represents the temperature of ice. The problem is then a model of the process of melting. In the condition (1.7), $\Phi \equiv 0$. The condition (1.8) reduces to the equation of conservation of energy:

$$\alpha_1 \, \partial u_1/\partial x - \alpha_2 \, \partial u_2/\partial x = ds_i/dt \tag{1.9}$$

where α_1, α_2 are positive (constants or functions). We further have, $g_1 \geqq 0$, $h_1 \geqq 0$, $f_1 \leqq 0$, $g_2 \leqq 0$, $h_2 \leqq 0$, $f_2 \geqq 0$. The literature on this problem is quite extensive. We refer to the books of Friedman [1] and Rubinstein [2] and to the references given there. Some more recent work can be found in [3]–[10]; see also Kyner [11] for a nonlinear Stefan problem. Much of the work is concerned with the one phase problem, that is, $k = 2$ and $u_2 \equiv 0$, $b_k = \infty$. In this case the free boundary $x = s_1(t)$ is a monotone function.

In most of the papers on the Stefan problem the authors reduce the problem to that of solving nonlinear integro-differential equations of Volterra type (for either $s_i(t)$, or $ds_i(t)/dt$, or $\partial u_j(s_i(t), t)/\partial x$). Then they proceed to prove the existence of a solution in a small time interval. In order to derive the existence of a global solution, it is necessary to obtain an a priori bound on the first derivatives $\partial u_i(x, t)/\partial t$ for $t<\tau$; the bound must be a bounded function of τ. Such a bound has been derived for the one phase problem [12]; for the k-phase problem this bound has been derived only for small data [6].

Results in the direction of problem (iii) are obtained from the proof of existence of the solution. Problem (ii) has been solved by Rubinstein [13] and Cannon and Hill [3], [4]. Problem (iv) for the one phase case was treated by Friedman [14] and Cannon and Hill [5]. For $k=2$ Rubinstein [13] established the asymptotic convergence of classical solutions; however, it is not known whether such solutions do indeed exist (except for small data).

The following problems remain unsolved by the above classical approach (of reduction to a nonlinear integro-differential equation) and by variants of it:

Problem A. Existence of a global solution for the k-phase Stefan problem, $k\geqq 2$.

Problem B. If the boundary data converge (in a suitable norm) as $t\to\infty$, does the solution have a limit as $t\to\infty$?

These problems have been solved in Friedman [15], where the concept of a solution is "nearly" classical. The methods depend however on techniques developed for the n-dimensional Stefan problem. We shall therefore consider first (in §2) the latter problem, and then (in §3) return to Problems A, B.

We conclude this section with a few references to non-Stefan problems in one dimension. In [16], [17] the free boundary problems differ from the Stefan problem in the form of the function Ψ (occurring in (1.8)). There is also a small parameter entering into the problem. It is shown that the (global) solution exists and that it has a power series expansion in α. Furthermore, asymptotic formulas are derived for the free boundary $x=s(t, \alpha)$ as $t\to\infty$, $\alpha\to 0$. Kruzhkov [18] has considered a one phase free boundary problem arising from the impact of a viscoplastic rod on a rigid obstacle. Here the conditions at the free boundary $x=s(t)$ are:

$$\partial u/\partial x = 0, \qquad u = A(s, t)$$

where A is some integral operator. He proved the uniqueness and the monotone behaviour of the solution. However, the existence of a solution is not known. Fulks and Guenther [19] have recently considered a two phase free boundary problem of two incompressible fluids in a compressible porous medium. Here the conditions at the free boundary are:

$$u_1 = u_2 = ds/dt, \qquad H + \gamma\, \partial u_1/\partial x = \lambda\, \partial u_2/\partial x$$

where H is some functional of u_1. They proved the existence and uniqueness of a global solution.

2. ***n*-dimensional Stefan problem.** We begin with a classical formulation of the problem. Let G be a bounded domain in R^n, whose boundary consists of connected manifolds M_i, $i=1, 2, \cdots, k_1$, $k_1 \geqq 2$. Let

$$\partial_1 G = \bigcup_{i=1}^{k_0} M_i, \qquad \partial_2 G = \bigcup_{i=k_0+1}^{k_1} M_i.$$

We assume that $\partial_1 G$ and $\partial_2 G$ are both nonempty and (for definiteness) that $\partial_2 G$ contains the outer boundary of ∂G. Write

$$G(t) = \{(x, t); x \in G\}, \quad \Omega_T = \bigcup_{0<t<T} G(t), \quad \partial_i G(t) = \{(x, t); x \in \partial_i G\}.$$

The free boundary will consist of a set $\Gamma = \{\Gamma(t),\ 0<t<T\}$ where $\Gamma(t)$ is a manifold lying in $G(t)$. Denote by $G_i(t)$ the open set bounded by $\partial_i G(t)$ and $\Gamma(t)$ and let

$$\Omega_{T,i} = \bigcup_{0<t<T} G_i(t).$$

Consider the parabolic equations

$$\alpha_1 \frac{\partial u_1}{\partial t} = \Delta u_1 + c_1 u_1 \quad \text{in } \Omega_{T,1} \qquad (c_1 \leqq 0), \tag{2.1}$$

$$\alpha_2 \frac{\partial u_2}{\partial t} = \Delta u_2 + c_2 u_2 \quad \text{in } \Omega_{T,2} \qquad (c_2 \leqq 0) \tag{2.2}$$

where α_1, α_2 are positive constants, the initial conditions

$$u_1(x, 0) = h_1(x) \quad \text{in } G_1(0) \qquad (h_1 \geqq 0), \tag{2.3}$$

$$u_2(x, 0) = h_2(x) \quad \text{in } G_2(0) \qquad (h_2 \leqq 0), \tag{2.4}$$

and the boundary conditions

$$u_1 = g_1 \quad \text{on } \partial_1 G \times (0, T) \qquad (g_1 \geqq 0), \tag{2.5}$$

$$u_2 = g_2 \quad \text{on } \partial_2 G \times (0, T) \qquad (g_2 \leqq 0). \tag{2.6}$$

Suppose $\Gamma(t)$ is given by $\Phi(x, t) = 0$. Then we impose the following conditions:

$$u_1 = u_2 = 0 \quad \text{on } \Gamma, \tag{2.7}$$

$$\alpha_1 \nabla_x u_1 \cdot \nabla_x \Phi - \alpha_2 \nabla_x u_2 \cdot \nabla_x \Phi = \alpha \Phi_t \quad \text{on } \Gamma \tag{2.8}$$

where α is a positive constant.

In general, when $n \geqq 2$, a classical solution (u_1, u_2, Φ) of the Stefan problem (2.1)–(2.8) does not exist. Indeed, a large portion of the free boundary may disappear all of a sudden when a thin island of ice is surrounded by warm water.

We proceed to define the concept of a weak solution. Set

$$\begin{aligned} g &= g_i/\alpha_i && \text{on } \partial_i G \times (0, T), \\ h &= h_i/\alpha_i && \text{on } G_i(0), \\ a(u) &= \alpha_1 u && \text{if } u > 0, \\ &= \alpha_2 u - \alpha && \text{if } u < 0. \end{aligned}$$

Let $u(x, t)$ be a measurable function in Ω_T. A bounded measurable function $\gamma(x, t)$ is denoted by $a(u(x, t))$ if

(i) at each point (x, t) where $u(x, t) \neq 0$, $\gamma(x, t) = a(u(x, t))$, and

(ii) at each point (x, t) where $u(x, t) = 0$, $-\alpha \leqq \gamma(x, t) \leqq 0$.

DEFINITION. A *weak solution* of (2.1)–(2.8) is a bounded measurable function $u(x, t)$ in Ω_T for which there is a function $a(u(x, t))$ such that the following is true:

$$\text{(2.9)} \qquad \begin{aligned} &\int\int_\Omega [u(\Delta\phi + c\phi) + a(u)\phi_t] dx dt \\ &\qquad = \int_0^T \int_{\partial G} g \frac{\partial \phi}{\partial \nu} dS_x dt - \int_{G(0)} a(h)\phi \, dx \end{aligned}$$

for any ϕ such that $\nabla_x \phi$, $\nabla_x^2 \phi$, ϕ_t are continuous in $\bar{\Omega}_T$ and $\phi = 0$ on $G(T)$ and on $\partial G \times (0, T)$. Here $c = c_1/\alpha_1$ at the points where $u > 0$ and $c = c_2/\alpha_2$ at the points where $u < 0$.

The above definition is due to Kamenomostskaja [20]. One can easily show that a classical solution is a weak solution. Conversely, a weak solution for which the null set is a smooth manifold and which is sufficiently smooth on each side of this manifold is a classical solution.

Using a finite difference scheme, Kamenomostskaja [20] proved the existence of a weak solution and the bound

$$\int\int_\Omega |\nabla_x u|^2 dx dt \leqq C.$$

In [21] Friedman used a simpler method based on energy inequalities and a "regularization" of the problem, to prove the following stronger result:

(R_1) Under some standard regularity assumptions on the data, there exists a weak solution u satisfying

$$\int_G |\nabla_x u(x, t)|^2 dx \leqq C \qquad (0 \leqq t \leqq T). \tag{2.10}$$

Kamenomostskaja [20] showed that the weak solution is unique. This is contained in the following stability theorem proved by Friedman [21]:

(R_2) Let u and $\hat{u}$ be two weak solutions corresponding to data h, g and $\hat{h}$, $\hat{g}$ respectively. Let Ψ and $\hat{\Psi}$ be any extensions of g and $\hat{g}$ into Ω_T. Then

$$\begin{aligned} &\int_0^T \int_G (\hat{u} - u)[a(\hat{u}) - a(u)]dxdt \\ &\leqq B \int_{G(0)} (\hat{h} - h)^2 dx + \int_0^T \int_G [(\hat{\Psi} - \Psi)^2 + |\nabla_x(\hat{\Psi} - \Psi)|^2]dxdt, \end{aligned} \tag{2.11}$$

where B is a constant independent of T (provided $c(x, t)$ is "well behaved" as $t \to \infty$).

Combining (2.10), (2.11) and using Sobolev type inequalities, the following result is obtained:

(R_3) Assume that $g(x, t)$ is convergent to $g_\infty(x)$ as $t \to \infty$, in the following sense:

$$\sup[|g| + |D_x g| + |D_t g| + H_\eta(D_x g)] < \infty$$

where $D_x g$ is the gradient of g restricted to $\partial G \times \{t\}$, and H_η is a Hölder coefficient for some exponent $\eta > 0$, and

$$\int_1^\infty \int_{\partial G} [(g_t)^2 + (g - g_\infty)^2 + |D_x(g - g_\infty)|^2] dS_x\, dt < \infty.$$

Then, for $n \geqq 2$ and $p < 2n/(n-2)$,

$$\int_G |u(x, t) - w(x)|^p dx \to 0 \quad \text{if } t \to \infty, \tag{2.12}$$

and, for $n = 1$,

$$u(x, t) \to w(x) \quad \text{as } t \to \infty, \quad \text{uniformly in } x \in G. \tag{2.13}$$

Here $w(x)$ is the harmonic function in G satisfying $w = g_\infty$ on ∂G.

The method of construction of the solution in [21] yields the following result:

(R_4) If $h \leqq \hat{h}$, $g \leqq \hat{g}$ then $u \leqq \hat{u}$.

DEFINITION. Let u be a weak solution. The set S where $-\alpha < a(u(x, t)) < 0$ is called the *generalized free boundary*. It is determined up to a set of measure zero.

If $h_2 \equiv 0$, $g_2 \equiv 0$ and if ∂G_2 is empty (that is, the ice has temperature $u_2 \equiv 0$ and it extends to ∞) then we call the Stefan problem a one phase problem. For such a problem it was proved in [21] that

$$\text{int}\, S = \emptyset. \tag{2.14}$$

It is still an open question whether or not

$$\text{meas.}\, S = 0. \tag{2.15}$$

As for the general n-dimensional Stefan problem, it is not known whether (2.14) is true.

In [21] the following asymptotic bounds are derived on the location of the generalized free boundary for the one phase problem:

(R_5) Assume that

$$\gamma < \frac{g_1(x, t)}{1 + t^{(n-2)/2}} < \gamma' \qquad \text{if } n \geqq 3,$$

$$\gamma < \frac{g_1(x, t)}{\log(2 + t)} < \gamma' \qquad \text{if } n = 2,$$

$$\gamma < g_1(x, t) < \gamma' \qquad \text{if } n = 1$$

where γ, γ' are positive constants. Then there exist positive constants β, β' such that if $(x, t) \in S$ then $\beta < |x| / t^{1/2} < \beta'$.

3. One dimensional Stefan problem with several phases. The results of §2 can be sharpened in the case $n = 1$. In particular, one can establish (see [21]) the continuity of the weak solution $u(x, t)$. A more detailed analysis in [15] shows that the generalized free boundary can be normalized so that in the k-phase problem the following is true:

(R_6) The generalized free boundary consists of a finite number of piecewise continuous curves $x = s_i(t)$. The points of (possible) discontinuity $\tau_1, \cdots, \tau_m$ are such that at each τ_j the number of phases strictly decreases. The solution u is a classical solution of the parabolic equations (for water or for ice) outside the curves $x = s_i(t)$. On these curves, $u = 0$ and the equation (1.9) is satisfied in an integrated form.

Thus we see that a global solution for the k-phase problem exists, and it is "nearly" classical. Problem A posed in §1 is thus essentially solved. As for Problem B, we have already stated that (2.13) holds. In addition, using the results of (R_6) one can also show that

(3.1) $\lim_{t \to \infty} s_i(t)$ exists.

Thus, Problem B is completely solved. There still remains the question of deriving bounds on the rates of convergence in (2.13) and in (3.1).

In proving (R_6), use is made of the existence of a classical solution for small time intervals for the k-phase problem.

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