MODULAR REPRESENTATIONS OF CLASSICAL LIE ALGEBRAS

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Let K be an algebraically closed field of prime characteristic p. By a classical Lie algebra over K we shall understand a Lie algebra $\mathfrak{g}$ obtained from a complex simple Lie algebra $\mathfrak{g}_C$ by the well-known procedure of Chevalley: see [7] [1], for example. In this note we announce some results on the representation theory of $\mathfrak{g}$ over K; proofs will appear elsewhere. All modules considered will be finite-dimensional and restricted, unless otherwise specified.

0. **Preliminaries.** Denote by Σ the root system of $\mathfrak{g}_C$ relative to a Cartan subalgebra, and let $\Pi = \{\alpha_1 \cdots \alpha_l\}$ be a simple system. Fix a Chevalley basis $\{X_\alpha, \alpha \in \Sigma; H_i, 1 \leqq i \leqq l\}$ of $\mathfrak{g}_C$; if $\mathfrak{g}_{\boldsymbol{Z}}$ is the $\boldsymbol{Z}$-span of this basis, then $\mathfrak{g} = \mathfrak{g}_{\boldsymbol{Z}} \otimes K$. For convenience, we also denote by X_α, H_i the corresponding elements of $\mathfrak{g}$. Write $\mathfrak{h} = \mathfrak{h}_{\boldsymbol{Z}} \otimes K$ (= span of the H_i in $\mathfrak{g}$). Kostant's theorem [7, §2] describes the $\boldsymbol{Z}$-form $\mathfrak{U}_{\boldsymbol{Z}}$ of the universal enveloping algebra of $\mathfrak{g}_C$ generated by all $X_\alpha^m/m!$ $(\alpha \in \Sigma, m \geqq 0)$.

If we let V_λ be the irreducible $\mathfrak{g}_C$-module of highest weight λ, and let $v_0 \in V_\lambda$ be a maximal vector (a nonzero vector annihilated by all X_α, $\alpha \in \Pi$), then $\mathfrak{U}_{\boldsymbol{Z}} v_0$ is an "admissible lattice." Tensoring with K yields a (restricted) $\mathfrak{g}$-module $\overline{V}_\lambda$, which is also a module for the simply connected Chevalley group G constructed from $\mathfrak{g}_C$ over K. If v_0 again denotes the maximal vector $v_0 \otimes 1$ in $\overline{V}_\lambda$, then v_0 has weight λ.

Let Λ denote the collection of p^l restricted weights λ characterized by the conditions $0 \leqq \lambda(H_i) < p$, $1 \leqq i \leqq l$. For each $\lambda \in \Lambda$ let M_λ be the irreducible $\mathfrak{g}$-module of highest weight λ; it is known that M_λ is a homomorphic, but not always isomorphic, image of $\overline{V}_\lambda$. The collection $\mathfrak{M} = \{M_\lambda | \lambda \in \Lambda\}$ exhausts the (isomorphism classes of) irreducible $\mathfrak{g}$-modules. Let $\mathfrak{U}$, $\mathfrak{H}$ be the restricted universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{h}$ over K (u-algebras). (Left) $\mathfrak{U}$-modules correspond precisely to restricted (left) $\mathfrak{g}$-modules. Every u-algebra is a Frobenius algebra, and $\mathfrak{U}$ is even symmetric.

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1. Standard cyclic modules and characters.

DEFINITION. A cyclic $\mathfrak{g}$-module, generated by a maximal vector (of weight λ), will be called standard cyclic (of weight λ).

PROPOSITION 1. *If* $\lambda \in \Lambda$, *the* $\mathfrak{g}$-*module* $\overline{V}_\lambda$ *is standard cyclic of weight* λ.

PROPOSITION 2 (BRADEN). *A standard cyclic* $\mathfrak{g}$-*module* (*restricted or not*) *is indecomposable and possesses a unique maximal submodule.*

In characteristic 0 the "most general" standard cyclic module for $\mathfrak{g}_C$ is always infinite-dimensional [6], [8], [9]. Here we consider the analogue for $\mathfrak{g}$. If $\{\beta_1, \cdots, \beta_m\}$ is the set of positive roots (relative to Π), let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_m$ be the corresponding X_{β_i} and $X_{-\beta_i}$, respectively. Let $\mathfrak{n}$, $\mathfrak{n}'$ be the subalgebras of $\mathfrak{g}$ spanned by the X_i, Y_i respectively, and let $\mathfrak{N}$, $\mathfrak{N}'$ be their u-algebras. If $\lambda \in \Lambda$, denote by I_λ the left ideal in $\mathfrak{U}$ generated by all X_i $(1 \leqq i \leqq m)$ and all $H_i - \lambda(H_i)\cdot 1$ $(1 \leqq i \leqq l)$. Set $Z_\lambda = \mathfrak{U}/I_\lambda$. The canonical map $\mathfrak{U} \rightarrow Z_\lambda$ induces a vector space isomorphism of $\mathfrak{N}'$ onto Z_λ: indeed, the coset of 1 in Z_λ is a maximal vector of weight λ, forcing dim $Z_\lambda \leqq p^m =$ dim $\mathfrak{N}'$, and on the other hand one can verify that $\mathfrak{N}' \cap I_\lambda = 0$. Moreover, any standard cyclic $\mathfrak{g}$-module of weight λ is a homomorphic image of this "universal" one.

Next we introduce certain "characters" analogous to those of Harish-Chandra [6, Exposé 19]. Let $\mathfrak{C}$ be the center of $\mathfrak{U}$. Since Z_λ is indecomposable (Proposition 2), Fitting's Lemma allows one to show that each $C \in \mathfrak{C}$ acts as a scalar plus a nilpotent; in particular, the function $\chi_\lambda: \mathfrak{C} \rightarrow K$ assigning to C its single eigenvalue on Z_λ, is a homomorphism of K-algebras. Moreover, $\chi_\lambda(C)$ is the single eigenvalue of C on any subhomomorphic image of Z_λ, from which we deduce:

PROPOSITION 3. $\chi_\lambda = \chi_\mu$ *if* M_λ, M_μ *occur as composition factors of some standard cyclic* $\mathfrak{g}$-*module.*

2. Linked weights and blocks.

DEFINITION. Let W be the Weyl group of $\mathfrak{g}_C$, $\rho =$ half-sum of positive roots. If λ, $\mu \in \Lambda$, viewed as functions on $\mathfrak{h}$, satisfy: $\lambda + \rho = (\mu + \rho)^\sigma$ for some $\sigma \in W$, then we say λ and μ are *linked* and write $\lambda \sim \mu$.

It is clear that linkage is an equivalence relation on Λ, since $(\lambda_\sigma)_\tau = \lambda_{\sigma\tau}$, where we write $\lambda_\sigma = (\lambda + \rho)^\sigma - \rho$. There is always a linkage class having only one member: take $\lambda = (p-1)\rho$; this weight yields the "Steinberg module" $M_\lambda = \overline{V}_\lambda = Z_\lambda$, the unique irreducible $\mathfrak{g}$-module of maximal dimension p^m. The condition $\lambda \sim \mu$ is analogous to Harish-

Chandra's condition for equality of "characters" in the infinite-dimensional case [6, Exposé 19].

THEOREM 1. *$\lambda \sim \mu$ implies $\chi_\lambda = \chi_\mu$.*

Although a precise description of the submodules of Z_λ is lacking, the following can be shown.

PROPOSITION 4. *$\lambda \sim \mu$ implies that Z_λ and Z_μ have the same composition factors (multiplicities counted). Up to scalar multiples, Z_λ has a unique minimal vector, namely, the coset of $Y_1^{p-1} \cdots Y_m^{p-1}$ (for any ordering of $Y_1, \cdots, Y_m$).*

The linkage class of λ is in 1-1 correspondence with the W-orbit of $\lambda+\rho$ in Λ, so Theorem 1 shows there are no more characters than orbits. We can relate this to the *blocks* of $\mathfrak{U}$ as well [4, §55]. The distinct (left) principal indecomposable modules (PIM's) of $\mathfrak{U}$ correspond 1-1 with the elements of $\mathfrak{M}$: The PIM U_λ has unique highest composition factor M_λ. Two PIM's are said to be "linked" if they share a composition factor, and the sum of all PIM's in a class of this equivalence relation is an indecomposable two-sided ideal of $\mathfrak{U}$, called a "block." Let B_λ be the block containing U_λ. It is easy to see that (under the canonical map $\mathfrak{U} \rightarrow Z_\lambda$) some copy of U_λ maps *onto* Z_λ, whence every composition factor of Z_λ belongs to the block B_λ. In view of Theorem 1 and Proposition 4, we can state:

THEOREM 1'. *$\lambda \sim \mu$ implies U_λ and U_μ are linked (so $B_\lambda = B_\mu$).*

This shows that the number t of distinct blocks does not exceed the number of W-orbits in Λ (and each block corresponds to a union of such orbits). Moreover, $t = \dim(\mathfrak{C}/\mathrm{rad}\ \mathfrak{C})$, and the χ_λ coincide with the homomorphisms $\mathfrak{C} \rightarrow K$ defined by the respective block idempotents [4, §85 and references].

3. **Invariants.** In order to prove the converse of Theorem 1 (under some restriction on p) it is necessary to examine more closely how $\mathfrak{C}$ acts on Z_λ. There is a natural K-linear map $\beta: \mathfrak{U} \approx \mathfrak{N}' \otimes \mathfrak{H} \otimes \mathfrak{N} \rightarrow \mathfrak{H}$ defined by $\beta(YHX) = 0$ if Y or X is not 1, $\beta(YHX) = H$ if $Y = X = 1$ ($Y \in \mathfrak{N}'$, $H \in \mathfrak{H}$, $X \in \mathfrak{N}$ standard basis monomials). If $\lambda \in \Lambda$ is viewed as a K-algebra homomorphism $\mathfrak{H} \rightarrow K$, then in view of the way χ_λ was defined, we have $\chi_\lambda(C) = \lambda(\beta(C))$, $C \in \mathfrak{C}$, and moreover, $\beta | \mathfrak{C}$ is multiplicative. Let γ be the K-algebra automorphism of $\mathfrak{H}$ sending H_i to $H_i - \rho(H_i)$ (ρ as before). Then Theorem 1 implies that $\gamma(\beta(C))$ lies in $\mathfrak{H}^W$ (= algebra of W-invariants in $\mathfrak{H}$), so $\dim \mathfrak{H}^W = t' \geqq t$. Now $\mathfrak{H}^W$ is a commutative semisimple associative algebra, and the corresponding t' K-algebra homomorphisms $\mathfrak{H}^W \rightarrow K$ are just the restric-

tions to $\mathfrak{H}^W$ of the $\lambda\in\Lambda$, those which are W-conjugate having the same restriction (so t' = number of W-orbits in Λ). To prove the converse of Theorem 1, it would suffice to prove that $t=t'$, or that $\gamma(\beta(C))=\mathfrak{H}^W$. This seems likely to hold in general, but our method, based on reduction mod p, does not work for "small" p.

THEOREM 2. *If $p>$ Coxeter number of Σ, then $\chi_\lambda=\chi_\mu$ implies $\lambda\sim\mu$.*

REMARK. The Coxeter number h (= order of the product of all simple reflections in W) for each of the simple types is as follows [2, pp. 250–275]: A_l, $l+1$; B_l, C_l, $2l$; D_l, $2l-2$; E_6, 12; E_7, 18; E_8, 30; F_4, 12; G_2, 6. If $p>h$, p does not divide the order of W.

4. **Projective modules.** We recall [4, §56] that the projective $\mathfrak{U}$-modules are just the direct sums of the PIM's (which are the only indecomposable projectives). It is easy to see that if M is indecomposable and $P\to M\to 0$ is a projective cover, then a sum of PIM's from the *same* block already maps onto M. Since every $\mathfrak{U}$-module has a projective cover, we deduce from Theorem 2:

THEOREM 3. *If $p>h$, then if M is an indecomposable $\mathfrak{U}$-module, all composition factors of M have highest weights which are linked.*

This has been conjectured in general by Verma; Pollack's study of type A_1, confirms the result directly [5], and Braden's conclusions [3] are highly consistent with it.

In [5] Pollack describes the PIM's for A_1 explicitly. For higher ranks we get some analogous results, the first of which resembles a classical theorem on group algebras of finite groups [4, 65.17].

PROPOSITION 5. *Every projective $\mathfrak{U}$-module is projective as $\mathfrak{N}'$-module; in particular, each PIM of $\mathfrak{U}$ has dimension divisible by p^m (m = number of positive roots).*

PROPOSITION 6. *If $\mathfrak{B}'$ is the subalgebra of $\mathfrak{U}$ generated by $\mathfrak{H}$ and $\mathfrak{N}'$, then every projective $\mathfrak{U}$-module is a projective $\mathfrak{B}'$-module. The PIM's of $\mathfrak{B}'$ are just the p^l modules Z_λ ($\lambda\in\Lambda$) regarded as $\mathfrak{B}'$-modules.*

The proof of Proposition 6 is a direct construction in $\mathfrak{U}$. Using this result, along with Proposition 4, one can get precise information about dimensions.

THEOREM 4. *Let C be the Cartan matrix of $\mathfrak{U}$ ($c_{\lambda\mu}$ = multiplicity of M_μ as composition factor of U_λ), and let D be the matrix $(d_{\lambda\mu})$, where $d_{\lambda\mu}$ = multiplicity of M_μ as a composition factor of Z_λ. Whenever the* *conclusion of Theorem 3 is valid,* $C={}^tD\cdot D$, dim $U_\lambda=a_\lambda d_{\lambda\lambda}p^m$ *and* dim $B_\lambda=a_\lambda p^{2m}$, *where* $a_\lambda=$ *cardinality of W-orbit of* $\lambda+\rho$ *in* Λ.

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