ALGEBRAIC COHOMOLOGY OF TOPOLOGICAL GROUPS

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In this note we discuss algebraic cohomology groups of topological groups. Eilenberg-Maclane [3] and Hopf [6] introduced the notion of algebraic cohomology of abstract groups, and definitions taking into account the topology are given in [2], [4], [5], and [8]. We give a definition which coincides with the usual one for discrete groups and generalizes those in [2], [5] and [8] for topological groups. It has the good functorial properties of being an "exact connected sequence of functors" in a suitable sense, and of being effaceable and universal.

All groups and modules considered will be Hausdorff.

The classical theory assigns to an abstract group G and a G-module A an exact connected sequence of functors $H^i(G, A)$ $(0 \leqq i < \infty)$. It can be shown that this sequence is universal and effaceable, and for any other effaceable exact connected sequence of functors $\check{H}$ with $\check{H}^0(G, A) \cong H^0(G, A)$ for all A, one has $\check{H}^i(G, A) \cong H^i(G, A)$ for all i by Buchsbaum's criterion.

If G is a topological group, a topological G-module A will mean an abelian topological group A with a jointly continuous action of G satisfying $g(a+a') = ga+ga'$, $(gg')a = g(g'a)$, $1a = a$. We show that topological G-modules form a quasi-abelian category in the sense of Yoneda [9], and define $H^i(G, A) = \mathrm{Ext}^i(\boldsymbol{Z}, A)$ where Ext is given by the definition of Yoneda for the quasi-abelian category of topological G-modules. $H^0(G, A)$ will then be the abstract group of points of A fixed under the action of G. If the underlying spaces of all groups and modules in question are limits of sequences of compact sets, we show that H^1 and H^2 have the obvious interpretations in terms of continuous crossed homomorphisms and extensions of topological groups.

Yoneda shows that with the appropriate definitions the Ext^i form an effaceable exact connected sequence of functors; one can further show that they are universal and prove a modified form of Buchsbaum's criterion. The "bar construction" semisimplicial resolution of an abstract group G [7] becomes a semisimplicial space in an obvious way if G is a topological group. If A is a G-module consider the sheaf of germs of A-valued functions on each space of this semisimplicial resolution. The canonical resolutions of these sheaves give rise to a

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double complex. $\hat{H}^i(G, A)$ will denote the ith cohomology group of this double complex. We then have:

THEOREM. *If G is locally compact, σ-compact and finite dimensional or a countable* CW-*complex and A is complete metric and either locally connected or locally compact, then $H^i(G, A) = \hat{H}^i(G, A)$.*

THEOREM. *Suppose G is as in the previous theorem. If A is discrete, $H^i(G, A)$ is isomorphic to the cohomology $H^i(B_G, A)$ of the classifying space B_G with coefficients in A.*

THEOREM. *If G is a connected Lie group, and A a finite dimensional vector space which is a differentiable G-module, then $H^i(G, A) \cong H^i_{\mathrm{Lie}}(\mathcal{G}, \mathcal{K}, A)$; the latter is the cohomology of the Lie algebra $\mathcal{G}$ of G modulo the Lie algebra $\mathcal{K}$ of a maximal compact subgroup.*

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