ON THE MINIMUM NORM PROPERTY OF THE FOURIER PROJECTION IN L^1-SPACES AND IN SPACES OF CONTINUOUS FUNCTIONS[1]

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Introduction. 1. Let $C(T)$ be the Banach space of complex continuous periodic functions on the real line, and $L^1(T)$ the Banach space of complex periodic functions on the real line which are absolutely integrable on $[0, 2\pi)$. For simplicity we shall sometimes denote both spaces by $E(T)$. Let then E_n be the space of trigonometric polynomials $\sum_{k=-n}^{+n} c_k e^{ikt}$, and let $F_n: E(T) \to E_n$ be the Fourier projection, defined by

$$(F_n x)(t) = \sum_{k=-n}^{+n} (x)_k e^{ikt}, \quad \text{where } (x)_k = \frac{1}{2\pi} \int_{-\pi}^{+\pi} x(t) e^{-ikt} dt.$$

Then F_n has minimum norm among the projections $E(T) \to E_n$, [10], [1]. Similar results hold when $E(T)$ is replaced by other Banach spaces of functions, [2], [6].

It has been proved recently that F_n is the unique minimum norm projection $C_R(T) \to E_n$, i.e. that $P = F_n$ if P is a projection $C_R(T) \to E_n$ and $\|P\| = \|F_n\|$, [3], [4]. We prove that F_n is the unique minimum norm projection $L^1(T) \to E_n$, and that neither result can be generalized very much.

It is possible to replace T by any compact abelian group G, the set $\{e^{ikt}: -n \leqq k \leqq +n\}$ of characters of T by any finite set $\{e_\gamma: \gamma \in N \subseteq \hat{G}\}$ of characters of G, and furthermore to consider the mapping $E(G) \to E_N$ given by $x \to x * k$, where $E(G) = C(G)$ or $L^1(G)$, $E_N =$ the linear hull of $\{e_\gamma: \gamma \in N\}$, and $k = \sum_{\gamma \in N} c_\gamma e_\gamma$, $0 \neq c_\gamma \in \mathbf{C}$. It is this generalization we have studied ([7], [8] and [9]); however,

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for simplicity's sake, we shall state here only the results when $k=\sum_{\gamma\in N} e_\gamma$ is the Dirichlet kernel, i.e. when the studied mappings are the projections $E(G)\to E_N$. Other kernels (the Fejér kernel, the de la Vallée-Poussin kernel) could be discussed with our more general results.

2. The minimum norm property of the Fourier projection, initially proved by S. Losinski (+Charchiladse and Nikolayev), follows from a relation of D. L. Berman [1], [2] which can be generalized to our more abstract setting ([6] or [7]). Assume thus that the Fourier projection $F_N: E\to E_N$ has minimum norm, the set of projections $S: E\to E_N$ with the same norm as F_N is then convex.

DEFINITION. This convex set of projections will be called C_k^∞ when $E=C(G)$, and C_k^1 when $E=L^1(G)$.

3. PROPOSITION. *The dimension of C_k^∞ is at least equal to the number of $\gamma\in\hat{G}$, $\gamma\notin N-N$, where the Fourier transform of $|k|$ vanishes.*

The proof is constructive, we illustrate the result showing that C_k^∞ can have infinite dimension. (This is the case when G is infinite and N a finite subset of a proper subgroup of its dual $\hat{G}$. For example $G=\prod_{q=1}^\infty G_q$ and $N=$any finite subset of $\hat{G}$.) Specifying G by T and combining then a slight extension of the result of [3] with ours, we can give a necessary and sufficient condition for uniqueness of F_N as a minimum norm projection $C(T)\to E_N$, as soon as the kernel k fulfills some requirements. These requirements are satisfied when $E_N=E_n$.

4. PROPOSITION. *The Fourier projection is the only minimum norm projection $L^1(G)\to E_N$ when the Dirichlet kernel k and its multiples are the only elements of E_N which vanish at the roots of k in G.*

The condition is satisfied by the classical Fourier projection.

We also show that the result cannot be generalized too much, but our results are not as good as when $E(G)=C(G)$. We can find groups G and finite sets $N\subseteq\hat{G}$ such that the dimension of C_k^1 is as large as we wish, the number of elements of N being even bounded, but we do not know whether this dimension can become infinite.

1. Statement of the main results.

DEFINITION. For $x\in L^1(G)$, $\sigma(x)$ will be the set of functions ϕ such that $|\phi(g)|\leq 1$ a.e. in G and $x(g)=\phi(g)|x(g)|$ a.e. in G.

The functions ϕ are elements of $L^\infty(G)$ and the particular function ϕ such that $\phi(g)=0$ a.e. where $x(g)=0$ will be denoted by sgn x.

We denote by gx the translate $gx(h)=x(h-g)$ of x and by $g\delta$ the unit mass concentrated at the point $g \in G$. Consistently with this, $g\sigma(x)$ will be the set $\{y : y = gz \text{ for } z \in \sigma(x)\}$. We then have

LEMMA. *Let S be a minimum norm projection $L^1(G) \to E_N$. Then for all $\phi \in \sigma(\bar{k})$ we have that $\langle S(g\delta), g\phi \rangle = \|S\| = \|F_N\|$ for almost all $g \in G$.*

We say that a mapping $L^1(G) \to L^1(G)$ is real, if it maps real functions onto real functions. We say that a subset N of $\hat{G}$ is symmetric, if $N = -N$.

COROLLARY. *If N is symmetric, every minimum norm projection $L^1(G) \to E_N$ is real.*

Note. We have proved analogous results for $C(G)$ in [6]. (See also [7].)

THEOREM. *If the kernel k is determined, up to a constant factor, as an element of E_N by its roots in G, then the Fourier projection $x \to x * k$ is the unique minimum norm projection $L^1(G) \to E_N$.*

EXAMPLE 1. Let G be the circle group T, and N the classical part $\{-n, -(n-1), \cdots, 0, \cdots, (n-1), n\}$ of $\hat{T} = Z$. Then the classical Fourier projection $x \to x * d_n$, $d_n(t) = \sum_{q=-n}^{+n} e^{iqt}$, is the unique minimum norm projection $L^1(T) \to E_n$.

EXAMPLE 2. Let G be again any compact abelian group, and N a finite subgroup (or one of its cosets) of $\hat{G}$. Then the Fourier projection $x \to x * k$, $k = \sum_{\gamma \in N} e_\gamma$, is the unique minimum norm projection $L^1(G) \to E_N$.

2. Let $E(G)$ be again $C(G)$ or $L^1(G)$ and C_k be C_k^∞ or C_k^1. The convex set C_k is a facet of the sphere with radius $\|F_N\|$ of the normed space $L(E(G); E_N)$: this facet consists of the tangent points of this sphere at the affine manifold

$$V_k = \{S : S = F_N + R, R \in L(E(G); E_N) \quad \text{and} \quad R(E_N) = \{0\}\},$$

i.e. the affine manifold of projections $E(G) \to E_N$.

DEFINITION. $\dim(C_k)$ will be the dimension of C_k in $L(E(G); E_N)$, i.e. the complex dimension of

$$V(C_k) = \text{the complex affine submanifold of } V_k \text{ generated by } C_k.$$

If N is symmetric, the facet C_k consists of real mappings (preceding corollary and note). In this case we let

$$V_r(C_k) = \text{the real affine manifold generated by } C_k.$$

DEFINITION [5, p. 180]. A point x of C_k is an interior point of C_k, when every straight line lying in $V_r(C_k)$ and going through x intersects C_k on a line segment containing x as an interior point.

Note. If N is symmetric, $E(G)$ and E_N are spanned by their real elements and furthermore C_k then consists of real mappings (preceding corollary and note). It is easy to check that C_k then has the same dimension over the complex field as over the real field, i.e. that the complex dimension of $V(C_k)$ is equal to the real dimension of $V_r(C_k)$.

NOTATIONS. We let

$\rho(k) = \{\operatorname{sgn} k\} \cup \{h\delta : k(h) = 0\}$,

$g\rho(k) = \{y : y = gz \text{ for } z \in \rho(k)\}$,

p_k^∞ = complex linear hull of

$\bigcup_{g\in G}$ [orth. proj. of $g\rho(k) \otimes g\delta$ in $E_{\hat{G}\sim N} \otimes E_{-N}$],

p_k^1 = complex linear hull of

$\bigcup_{g\in G}$ [orth. proj. of $g\delta \otimes g\rho(k)$ in $E_{G\sim N} \otimes E_{-N}$].

THEOREM. *If G is a finite abelian group and N is symmetric, then*:

$$\dim (C_k^\infty) = ((\#G) - (\#N))(\#N) - \dim (p_k^\infty).$$

$$\dim (C_k^1) = ((\#G) - (\#N))(\#N) - \dim (p_k^1).$$

Furthermore the Fourier projection F_N is an interior point of the facet C_k.

COROLLARY 1. *If furthermore the kernel $k = \sum_{\gamma\in N} e_\gamma$ has no zeros in G, then* $\dim(C_k) \geqq (\#N)((\#G) - (\#N)) - (\#G)$, *which becomes arbitrarily large whenever $2 \leqq (\#N) \leqq a < \infty$ and $(\#G) \to \infty$.*

COROLLARY 2. *Given a positive integer $a \geqq 2$, a sequence of cyclic groups G can be chosen such that* $\dim(C_k) \to \infty$, *whenever we keep N symmetric, in arithmetic progression in $\hat{G}$, and such that $2 \leqq (\#N) \leqq a$.*

3. In this paragraph we let again $E(G) = C(G)$ or $L^1(G)$. We also let $G = G_1 \times G_2$, where G_1 and G_2 are compact abelian groups. Finite direct products can then be handled by finite induction.

DEFINITION [11]. A norm ν on the tensor product $E \otimes F$ of the normed spaces E and F will be called a crossnorm, whenever

$$(\forall x \in E)(\forall y \in F). \qquad \nu(x \otimes y) = \|x\|_E \|y\|_F.$$

Now let F_1 be a normed subspace of $E(G_1)$, F_2 a normed subspace of $E(G_2)$, and $F_1 \overline{\otimes} F_2$ the closure of $F_1 \otimes F_2$ in $E(G)$. Let s be a bounded linear mapping $E(G_1) \to F_1$, and t a bounded linear mapping $E(G_2) \to F_2$. We then establish that $s \otimes t$ defines a unique bounded linear mapping $E(G) \to F_1 \overline{\otimes} F_2 \subseteq E(G)$ by

$$(\forall \sum x_j \otimes y_j \in E(G_1) \otimes E(G_2)).\ (s \otimes t)(\sum x_j \otimes y_j) = \sum_j s(x_j) \otimes t(y_j).$$

This allows us to prove the following theorem, which can also be established for tensor products with suitable crossnorms of abstract Banach spaces.

THEOREM. *$L(E(G_1); F_1) \otimes L(E(G_2); F_2)$ is naturally isomorphic to a vector subspace of $L(E(G); F_1 \overline{\otimes} F_2)$, and the induced norm on this subspace is a crossnorm. Furthermore if the group G_1 (or G_2) is finite, and the normed subspace F_2 (or F_1) complete, than this vector subspace contains the elements of finite rank of $L(E(G); F_1 \overline{\otimes} F_2)$.*

If $G = G_1 \times G_2$, the natural projection of G onto G_i, $i = 1$ or 2, allows an identification of a space of functions on G_i with a space of functions on G. For this reason the closure $\overline{F}_1$ of F_1 in the complete space $E(G_1)$ is also a closed subspace of $E(G)$. We then have

COROLLARY 1. *Every bounded linear mapping $E(G_1) \to F_1$ has at least one extension with the same norm to $E(G) \to \overline{F}_1$.*

Now let N_1 and N_2 be respective finite subsets of $\hat{G}_1$ and $\hat{G}_2$. We then have

COROLLARY 2. *The tensor product of a minimum norm projector $E(G_1) \to E_{N_1}$ with a minimum norm projector $E(G_2) \to E_{N_2}$ is a minimum norm projector $E(G) \to E_{N_1+N_2}$.*

Now, denoting by C_N (instead of C_k) the convex facet of minimum norm projectors $E(G) \to E_N$, we have

COROLLARY 3. $\dim(C_{N_1+N_2}) \geqq \dim(C_{N_1}) + \dim(C_{N_2}) + \dim(C_{N_1}) \cdot \dim(C_{N_2})$.

COROLLARY 4. *Given a positive integer $a \geqq 2$, a sequence of direct products $G = G_1 \times G_2$, G_1 = cyclic group and G_2 = arbitrary compact abelian group, can be found such that $\dim(C_{N_1+N_2}) \to \infty$, whenever N_2 is an arbitrary finite subset of $\hat{G}_2$ and we keep N_1 symmetric, in arithmetic progression in $\hat{G}_1$, and such that $2 \leqq (\# N_1) \leqq a$.*

4. In this last paragraph we handle only $C(G)$, G being a compact abelian group. Let A_k be the symmetric set $\{\gamma \in \hat{G} : \gamma \notin N - N$ and $(|k|)_\gamma = 0\}$. Then

THEOREM. $\dim(C_k^\infty) \geqq$ *Cardinal of A_k. More precisely the real parts and the imaginary parts of the characters e_γ, $\gamma \in A_k$, yield a set of linearly independent mappings $R_\gamma : x \to (x(\mathrm{Re}\, e_\gamma)) * k$, $R_{-\gamma} : x \to (x(\mathrm{Im}\, e_\gamma)) * k$,*

$R_\gamma(E_N)=R_{-\gamma}(E_N)=\{0\}$, such that the projections $S_\gamma=F_N+R_\gamma$, $S_{-\gamma}=F_N+R_{-\gamma}$ are all minimum norm projections $C(G)\to E_N$.

Note. If the set A_k is infinite, one can of course find a set of algebraically linearly independent vectors α_ν, which has the power of the continuum and which is in the closed convex hull of $\{e_\gamma:\gamma\in A_k\}$. It follows from the proof of the preceding theorem that these vectors α_ν can be chosen such as to define linearly independent elements of the facet C_k^∞, which has hence a dimension equal to the power of the continuum. Furthermore the Fourier projection F_N is an interior point of this facet C_k.

Combining a particular case of this theorem with a slight extension of results of [3], we get a criterion for uniqueness of F_N as a minimum norm projection $C(\boldsymbol{T})\to E_N$, whenever the kernel k has a special form.

DEFINITION. A point g of the circle group $\boldsymbol{T}$ will be called an alternating point of a real kernel $k\in E_N$, whenever k (vanishes and) changes sign at g.

N.B. The Dirichlet kernel k is real if and only if N is symmetric.

COROLLARY. *Assume $G=\boldsymbol{T}$, N symmetric, and the Dirichlet kernel k determined, up to a constant factor, by its alternating points. Then the Fourier projection $F_N:x\to x*k$ is the unique minimum norm projection $C(\boldsymbol{T})\to E_N$ if and only if the symmetric set* $A_k=\{\gamma\in\hat{\boldsymbol{T}}:\gamma\notin N-N$ and $(|k|)_\gamma=0\}$ *is empty. Furthermore* $\dim(C_k^\infty)\geqq$ *Cardinal of A_k.*

EXAMPLE 1. G is a compact abelian group and N a finite subset of a proper subgroup Λ of $\hat{G}$ or of one of the cosets of Λ. Then

$$\dim(C_k^\infty)\geqq(\#\Lambda)((\#\hat{G}/\Lambda)-1).$$

In particular $\dim(C_k^\infty)=\infty$ if G is an infinite (separated) group.

PARTICULAR CASE OF EXAMPLE 1. Let $G=\prod_{q=1}^\infty G_q$, G_q = compact abelian group for each q, and let N be a finite subset of $\hat{G}=\bigotimes_{q=1}^\infty\hat{G}_q$. Then $N\subseteq\bigotimes_{q=1}^\alpha\hat{G}_q$ for some finite α, i.e. N is a finite subset of the proper subgroup $\bigotimes_{q=1}^\alpha\hat{G}_q$ of $\hat{G}$, and hence it follows from the preceding that $\dim(C_k^\infty)=\infty$.

EXAMPLE 2. In this last example we leave the class of projectors in order to study the Fejér kernel (cf. introduction). Let G be the circle group $\boldsymbol{T}$ and the kernel k be the Fejér kernel ϕ_n defined by:

$$\phi_n=\frac{1}{n}\sum_{q=0}^{n-1}k_q,\quad\text{where } (k_q)(t)=\sum_{j=-q}^{+q}e^{ijt},$$

i.e. k_q = Dirichlet kernel of order $(2q+1)$.

We let $S_{\phi_n} = x \to x*\phi_n : C(T) \to E_N$, where

$$N = \{-(n-1), \cdots, 0, \cdots, (n-1)\} \subseteq T = Z,$$

and s_{ϕ_n} = the restriction of S_{ϕ_n} to E_N.

We know that $x*\phi_n \to_n x$ in $C(T)$, and that $\phi_n \geqq 0$, i.e. $|\phi_n| = \phi_n$. Hence $(|\phi_n|)_\gamma = 0$ for $\gamma \notin N$. It then follows from the previous theorem (put into its more general form, cf. introduction) that $\dim(C^\infty_{\phi_n}) = \infty$, more precisely S_{ϕ_n} is the center of an infinite dimensional facet of minimum norm extensions $C(T) \to E_N$ of s_{ϕ_n}.

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