ON GROUP ALGEBRAS

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For any (discrete) group G and any field F, let FG denote the group algebra of G over F. Thus elements of FG are finite formal sums $\sum a(g)g$, where $a(g) \in F$, $g \in G$.

THEOREM 1. *Suppose G is any group and F is the field of complex numbers. Let $*$ denote the involution on FG given by*

$$\left(\sum a(g)g\right)^* = \sum \overline{a(g)}g^{-1}.$$

Let Z denote the center of FG. Then there exists a function $\natural$ from FG into Z with the following properties:

(i) $(a+b)^\natural = a^\natural + b^\natural$,
(ii) $(za)^\natural = za^\natural$ *for* $z \in Z$,
(iii) $(ab)^\natural = (ba)^\natural$,
(iv) $z^\natural = z$ *for* $z \in Z$,
(v) $(a^*)^\natural = (a^\natural)^*$,
(vi) $(aa^*)^\natural = 0$ *implies* $a = 0$.

In fact, if $W(G)$ is the W^*-algebra generated by the left action of G on $l^2(G)$, then we may embed FG in $W(G)$, and then the above function is just the restriction to FG of the function $\natural$ on $W(G)$ studied by Dixmier [2].

A two-sided ideal of a ring R is said to be an annihilator ideal of R if it is the left annihilator of some subset of R.

LEMMA 2. *Let F be a field of characteristic zero and G any group. If I is an annihilator ideal of FG and a, b are elements of I such that $axb - bxa \in I$ for all $x \in FG$, then there exist elements y, z in the center of FG, and not both zero, such that $ya - zb \in I$. If further $a(FG)b \not\subseteq I$, then y and z may be chosen so that $ya \notin I$.*

COROLLARY 3. *If F is algebraically closed of characteristic zero and FG is an order in a ring Q, then the center of FG is an order in the center of Q.*

Given a group G, let Δ denote the subgroup of elements which have only finitely many conjugates in G. Let Δ^+ denote the subgroup of torsion elements of Δ. By a result of Passman [5], FG is semiprime

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if and only if G has no finite normal subgroups of order divisible by the characteristic of F. By a result of Connell [1], FG is prime if and only if G has no finite normal subgroups.

THEOREM 4. *If FG is semiprime and has a classical ring of quotients which is finite dimensional over its center, then* $[G:\Delta]<\infty$.

From Theorem 4 and results of Procesi [7], Posner [6] and, Kaplansky [4] on polynomial identities, we obtain a strengthening of two special cases of a result of Isaacs and Passman [3].

COROLLARY 5. *If G has no finite normal subgroups and G satisfies a polynomial identity of degree n, then G has an abelian normal subgroup of finite index* $\leqq [n/2]^2$.

COROLLARY 6. *If G is finitely generated, FG is semiprime, and FG satisfies a polynomial identity, then G has an abelian subgroup of finite index.*

Also as an application of Lemma 2 we obtain

THEOREM 7. *If $F\subset K$ are fields of arbitrary characteristic and I is a prime ideal of KG which contains no nonzero central elements, then G has no finite normal subgroups and* $I\cap FG\neq 0$.

As an outgrowth of the investigations of annihilator ideals and rings of quotients of group algebras, the following results were obtained.

THEOREM 8. *Suppose FG is semiprime. Then if I is any annihilator ideal of FG,* $I=(I\cap F\Delta^+)FG$.

COROLLARY 9. *Suppose FG is semiprime, I an annihilator ideal of FG, and $a\in I$. Then there exists a central idempotent $e\in I$ such that* $a=ea$.

COROLLARY 10. *Suppose FG is semiprime. Then the classical ring of quotients of the center of FG is a von Neumann regular ring.*

Proofs of these results will appear in the author's doctoral thesis.

The author wishes to thank Professor D. S. Passman for pointing out that the technique of Theorem 4 could be applied in characteristic not zero, and that the idempotent produced in Corollary 9 is central. Professor Passman has recently developed the author's approach to polynomial identities in group algebras in a combinatorial rather than ring-theoretic manner to extend the results of I. M. Isaacs and himself [3] to the general semiprime case. He has informed the author that he also has a proof of Theorem 8.

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