GROMOLL GROUPS, Diff S^n AND BILINEAR CONSTRUCTIONS OF EXOTIC SPHERES

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1. **Introduction and main results.** The Kervaire-Milnor group Γ^n has a filtration by subgroups,

$$0 = \Gamma^n_{n-1} \subset \cdots \subset \Gamma^n_{k+1} \subset \Gamma^n_k \subset \cdots \subset \Gamma^n_1 = \Gamma^n,$$

due to Gromoll [9], which we study by means of certain homomorphisms

$$\pi_p(SO_q) \otimes \pi_q(SO_p) \xrightarrow{\ \sigma_{p,q}\ } \Gamma^{p+q+1} \xleftarrow{\ \tau_{p+1,q}\ } \Gamma^{p+1} \otimes \pi_q(SO_p)$$

See [12] for definitions. The pairing σ was first introduced by Milnor [13] and has been studied in [3], [11]. The pairing τ has been studied in [8], [16].

The groups of Gromoll are related to the homotopy groups of Diff S^n by a simple pasting construction: namely, there are homomorphisms $\lambda_i : \pi_i(\text{Diff } S^n) \to \Gamma^{n+i+1}$ with image $\lambda_i = \Gamma^{n+i+1}_{i+1}$ (see Proposition 2.1 and also [9, §1]).

We shall detect nontrivial elements in some Γ^n_{k+1}. Note that $\Gamma^n_{k+1} \neq 0$ implies that $\Gamma^n_{i+1} \neq 0$ and, hence, $\pi_i(\text{Diff } S^{n-i-1}) \neq 0$, for all $i \leqq k$. For slightly sharper statements see Proposition 3.3 and Proposition 3.4.

1.1. THEOREM. (a) $\Gamma^{4k-1}_{2k-2} \neq 0$, *for all* $k \geqq 4$.
(b) $\Gamma^{4k+1}_{2v(k)} \neq 0$, *for all* $k \geqq 0$, $k \neq 2^l - 1$.

Here $v(k)$ is the maximum number of linearly independent vector fields on S^{2k+1}. It is well known that $v(k) = 1$ when k is even and $v(k) \geqq 3$, when k is odd. Its precise value is given in [2].

Theorem 1.1 follows from some of our results on σ. Corollary 3.5, below, also based on work with σ, actually establishes fairly large lower bounds for the order of Γ^{4k-1}_{2k-2} (with some restrictions on k).

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1.2. THEOREM. (a) *Let Q be an odd prime, and let u and v be integers satisfying* $0 \leqq v < u \leqq Q-1$, $u-v \neq Q-1$. *Write* $n = 2(uQ+v+1)(Q-1) -2(u-v)-1$. *Then,* $\Gamma^n_{2Q-2} \supseteq Z_Q$. (b) Γ^9_2 *and* Γ^{10}_2 *are nontrivial.*

Theorem 1.2 is proved using τ (see Proposition 3.2). It generalizes results in [16].

1.3. THEOREM. Diff S^n *cannot be dominated by a finite* CW *complex, provided* $n \geqq 7$.

In particular, for this range of values of n, Diff S^n is not dominated by a finite-dimensional Lie group. This answers a question raised by J. Eells and R. Palais.

Theorem 1.3 contrasts with the fact that, for $n=1$, 2, Diff S^n has the homotopy type of SO_{n+1} [18]. The only undecided dimensions, therefore, are $n=3, 4, 5, 6$.

In §2 we deduce Theorem 1.3 from Theorem 1.1 (a) and Theorem 1.2 (b). In §3, we describe our results on σ and τ and give a table of low-dimensional computations. In §4, we relate our results to the inertia groups $I(\Sigma^n \times S^p)$, and we comment on Gromoll's pinching numbers δ_n.

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2. **Proof of Theorem 1.3.** Diff S^n (resp., Diff(S^n, D^n_+)) is the group of all C^∞, orientation-preserving diffeomorphisms of S^n (resp., those which keep fixed the upper hemisphere D^n_+). Give it the C^∞ topology. SO_{n+1} is a closed subgroup of Diff S^n. It is well known ([7], [17]) that Diff S^n and Diff(S^n, D^n_+) have the homotopy type of countable CW complexes and that the map $SO_{n+1} \times \text{Diff}(S^n, D^n_+) \to \text{Diff } S^n$ defined by group-multiplying the inclusions

$$SO_{n+1} \subset \text{Diff } S^n \supset \text{Diff}(S^n, D^n_+)$$

is a homotopy equivalence.

2.1. PROPOSITION. (a) *The multiplication of* Diff(S^n, D^n_+) *is homotopy-abelian.*

(b) *Let* $\lambda_i : \pi_i(\text{Diff } S^n) \to \Gamma^{n+i+1}$ *be as in* §1, *and let* μ_i *be its restriction to the direct summand* $\pi_i(\text{Diff}(S^n, D^n_+))$. *Then image* $\mu_i = \Gamma^{n+i+1}_{i+1}$.

Let $A_n = \text{Diff}(S^n, D^n_+)$ and note that $\pi_1 A_n = H_1 A_n$.

2.2. PROOF OF THEOREM 1.3. If Diff $S^n \sim SO_{n+1} \times A_n$ is dominated by a finite CW complex, for some n, then so is A_n, and so $H_*(A_n; Z_p)$ is finitely-generated, for all primes p. According to Browder [6], therefore, H_*A_n has no torsion. In particular, $\pi_1 A_n = H_1 A_n$ is free-abelian. Thus, the projective class group $\tilde{K}_0(\pi_1 A_n)$ vanishes, so that A_n has the homotopy type of a finite CW complex (Wall [21]). It now follows from Hubbuck [10] that the identity component of A_n has the homotopy type of a point or of a product of circles, so that $\pi_i A_n = 0, i \geqq 2$.

Theorem 1.1 (a) and Theorem 1.2 (b), together with Proposition 3.2 and the subsequent remark, imply that $\pi_1 A_7$ and $\pi_1 A_8$ have elements of finite order and that, for $n \geqq 9$, there is some $i \geqq 2$ such that $\pi_i A_n \neq 0$. Thus, $n \leqq 6$ as desired. This completes our proof.

Note that when $\pi_1 A_n$ has elements of finite order Browder's theorem alone implies that Diff S^n is not finitely dominated. Our results on the τ-pairing (Theorem 1.2 and Proposition 3.2) yield infinitely many such n, but not enough to prove Theorem 1.3.

3. **The pairings σ and τ.** The Gromoll groups are related to σ and τ by the next two propositions. Let $\mu_i : \pi_i(\mathrm{Diff}(S^n, D^n_+)) \to \Gamma^{n+i+1}$ be as in Proposition 2.1 (b).

3.1. PROPOSITION. *For any a, b, $0 \leqq a \leqq q$, $0 \leqq b \leqq p$, let $i_a : \pi_p(SO_{q-a}) \to \pi_p(SO_q)$ and $i_b : \pi_q(SO_{p-b}) \to \pi_q(SO_p)$ be the homomorphisms induced by the standard inclusions. Write $c = a+b+1$. Then, there is a homomorphism*

$$g_c : \pi_p(SO_{q-a}) \otimes \pi_q(SO_{p-b}) \to \pi_c(\mathrm{Diff}(S^{p+q-c}, D_+^{p+q-c}))$$

such that $\mu_c g_c = \sigma_{p,q} \circ (i_a \otimes i_b)$.

In particular, image $(\sigma_{p,q} \circ (i_a \otimes i_b)) \subset \Gamma^{p+q+1}_{c+1}$.

3.2. PROPOSITION. *For every $q > 1$, there is a homomorphism*

$$h_q : \Gamma^{p+1} \otimes \pi_q(SO_p) \to \pi_q(\mathrm{Diff}(S^p, D^p_+))$$

such that $\mu_q h_q = \tau_{p+1,q}$.

In particular, image $\tau_{p+1,q} \subset \Gamma^{p+q+1}_{q+1}$.

REMARK. Note that domain $\tau_{p+1,q}$ is finite, so that if image $\tau_{p+1,q} \neq 0$, then $\pi_q(\mathrm{Diff}(S^p, D^p_+))$ has elements of finite order.

To prove Theorem 1.2, we follow Novikov [16] and map $\tau_{p+1,q}$ into the composition pairing in stable homotopy. Then we apply results of Toda [19], [20].

The nonzero elements in Theorem 1.1 (b) are Kervaire spheres (which, of course, come from σ). We prove Theorem 1.1 (a), for large k, by applying the Eells-Kuiper μ-invariant, as in [11], to Milnor's plumbing construction [13] and by using the Barratt-Mahowald Splitting Theorem to show that μ takes the same values on image $\sigma_{4r-1,4s-1} \circ (i_a \otimes i_b)$ as on the entire image $\sigma_{4r-1,4s-1}$, provided $4s-1-a$

$>\max(2r, 12)$, and $4r-1-b>\max(2s, 12)$. For small k, we use Milnor's method [13] applied to the μ-invariant.

For sharper results on σ, we generalize some work of D. R. Anderson [3] and again apply the Barratt-Mahowald Splitting Theorem. To describe our conclusions, let

$$j_m = \text{order image } J_{4m-1} \quad \text{and} \quad b_m = (2^{2m-1} - 1)B_m a_m j_m/2m,$$

where B_m is the mth Bernoulli number, and $a_m = 1$ or 2, according as m is even or odd. Write

$$\rho_{r,s} = b_{r+s}/\text{g.c.d.}(b_{r+s}, b_r b_s).$$

3.3. PROPOSITION. *Let r and s be integers satisfying $r \geqq 6$, $s \geqq 6$, $r<2s<4r$, and write $t=r+s$. Then, $\Gamma^{4t-1}_{2t-2} \cap bP_{4t}$ contains a cyclic group of order $\rho_{r,s}$.*

3.4. PROPOSITION. (a) *Let r, s, t be as in 3.3. Then $\rho_{r,s}$ is odd and*

$$\rho_{r,s} > \tfrac{1}{8}(2t-1)\binom{2t-2}{2r-1} j_t/j_r j_s.$$

(b) *Write $r = 2^d(2e+1)$. Then $\rho_{r,r} > 2^{2r-d-9}$.*

REMARKS. The lower bound $\frac{1}{8}(2t-1)\binom{2t-1}{2r-1}j_t/j_r j_s$ is often large. For example, if r and s are primes, $7 \leqq r < s < 2r$. Then, this bound is larger than $2^{3r+s-8}/(2r+1)(2s+1)$. Much stronger but more complicated statements are possible.

When $r=s$, Proposition 3.3 is essentially Anderson's Theorem 1, [3], combined with Proposition 3.1. The proof of 3.4 involves complicated but elementary number theory.

We now display some divisors of Γ^n_k, k and n small. Results of [14], [15], [19], [20] are used for some of the calculations. Recall that $\Gamma^n_1 = \Gamma^n$ and $\Gamma^n_{k+1} \subset \Gamma^n_k$. According to Cerf, $\Gamma^n_2 = \Gamma^n$, for all n. For the reader's convenience, we give the order of $\Gamma^n_2 = \Gamma^n_1 = \Gamma^n$ precisely.

$k \backslash n$	13	15	19	21	22
2	3	16,256	523,264	4	4
3	3	4,064	2,044	2	2
4	3	2,032	2,044	2	2
5		1,016	2,044	2	
6		508	1,022	2	
7			511		
8			511		

Some divisors of order (Γ^n_k)

When entries are omitted for $n \leqq 22$, this means that our techniques give no additional information.

4. Remarks on $I(\Sigma^{p+1} \times S^q)$ and the Gromoll numbers δ_n.

4.1. *$I(\Sigma \times S^q) \subset \Gamma_{q+1}^{p+q+1}$, for all $\Sigma \in \Gamma^{p+1}$ and $q \geqq 2$.*

This follows from 3.2 and DeSapio's results on the τ-pairing [8].

4.2. *When $p \geqq 2q-1$, some $I(\Sigma^{p+1} \times S^q)$ have elements of odd prime order.*

This follows from Theorem 2.1 and DeSapio [8], and it contrasts with the fact, deducible from [4], that $I(\Sigma^{p+1} \times S^q)$ is 2-primary when $p < 2q-1$.

4.3. *There are spheres in image σ which are not in image τ.*

This follows from the last assertion in 4.2, together with 3.3 and 3.4 (a).

4.4. In [9], Gromoll defines an increasing sequence of real δ_k satisfying $\delta_1 = 1/4$ and $\lim \delta_k = 1$. He proves that *if the sphere Σ^n can be δ_k-pinched, then $\Sigma^n \in \Gamma_k^n$.* Since $\Gamma_{n-2}^n = 0$, [18], Σ^n can be δ_{n-2}-pinched only if Σ^n is diffeomorphic to S^n.

Question 1. *Can every sphere in Γ_k^n be δ_k-pinched?*

This probably asks too much, since no examples of riemannian exotic spheres admitting positive sectional curvature are known.

Call δ *N-universal* if $0 < \delta < 1$ and if Σ^n δ-pinched and $n \geqq N$ imply Σ^n diffeomorphic to S^n.

Question 2. *Does an N-universal δ exist, for some N?*

Question 2 was asked by Gromoll [9].

We simply remark here that an affirmative answer to either question implies a negative answer to the other, because $\Gamma_{2k-2}^{4k-1} \neq 0$, $k \geqq 4$.

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