ON PARALLELISM IN RIEMANNIAN MANIFOLDS

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The definition of parallelism along a curve in a Riemannian manifold extends to higher dimensional submanifolds. This note is to announce a local existence and uniqueness theorem, Theorem B(p), for the extended definition. A proof of the theorem in the C^∞ category will appear in [2]. A proof, in the C^ω category, under somewhat weaker conditions, will appear in [1]. A global C^∞ version under stronger assumptions appears in [3]. This note ends with a sketch of a new proof of Theorem B(p).

Let $g: N^p \to M^m$ be a (not necessarily isometric) smooth (that is, C^∞ or C^ω) immersion of Riemannian manifolds. Let E be a euclidean vector bundle over N and F a euclidean vector bundle over M. A vector bundle map $G: E \to F$ is a *vector bundle isometry along* g provided that G sends the fibers $E(n)$ isometrically into the fibers $F(g(n))$. When E and F are the tangent bundles ($T(N^p)$ and $T(M^m)$), G is called a *tangent bundle isometry* (T.B.I.) *along* g. The *normal bundle to a* T.B.I. G is the $m-p$ dimensional vector bundle $G^\perp$ over N whose fiber over $n \in N$ is the orthogonal complement $\perp G(N_n)$ to $G(N_n)$ in $M_{g(n)}$. The *second fundamental form of* G, $\mathrm{II}_G: G^\perp \to \mathrm{Hom}(T(N), T(N))$ is a vector bundle map defined as follows. Given $v \in \perp G(N_n)$ and $x, y \in N_n$ extend y to a vector field Y on N in some neighborhood of n, let ∇ be the covariant derivation on M and put

$$\langle \mathrm{II}_G(v)x, y\rangle_n = -\langle \nabla_{Tg(x)} G(Y), v\rangle_{g(n)}.$$

The definition is independent of the choice of Y.

G is *parallel along* g if $(\mathrm{trace}) \cdot \mathrm{II}_G: G^\perp \to R$ vanishes identically. It was shown in [1] that this definition is a generalization to higher dimensional immersed submanifolds, of the classical notion of parallelism along a curve. The significant facts are the following.

Every unit vector field along a curve $g: N^1 = (a, b) \to M$ corresponds in a natural way to a T.B.I. along g. Under this correspondence, parallel vector fields are paired with parallel T.B.I.'s.

An immersion $g: N^p \to M^m$ is isometric if and only if its tangent map

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$Tg:TN \to TM$ is a T.B.I. In such a situation, g is a minimal immersion if and only if Tg is parallel along g. Thus for every p, $1 \leqq p < m$, the critical manifolds of the calculus of variations problem for minimal p dimensional "area" are exactly the p dimensional autoparallels (i.e. the isometric immersions whose tangent maps are parallel).

Below, the same letter is used to designate a distribution on a manifold and the subbundle of the tangent bundle that it determines. If E is a vector bundle over Y and $i:X \to Y$ is a smooth map then $i_*:i^*E \to E$ is the induced map of the induced bundle.

THEOREM B(p). *Let $g:N^p \to M^m$ be an (not necessarily isometric) immersion of Riemannian manifolds. Let H be a $(p-1)$ dimensional distribution on N^p and (N^{p-1}, i) a homeomorphically embedded integral manifold of H. Suppose there is given as initial data:*

1. *$G^{p-1}:H \to T(M)$, a vector bundle isometry along g, and*
2. *$G^p:i^*T(N^p) \to T(M)$, a vector bundle isometry along $g \cdot i$.*

It is assumed that G^{p-1} and G^p are compatible:

$$G^p\big|_{i^*H} = G^{p-1} \cdot i_*: i^*H \to T(M).$$

Then, if the data is all C^ω, there is a neighborhood U of N^{p-1} in N^p and a unique parallel C^ω T.B.I. $G:T(U) \to T(M)$ that extends the initial data:

$$G\big|_H = G^{p-1}: H \to T(M) \text{ along } g\big|_U \quad \text{and}$$
$$G \cdot i_* = G^p: i^*T(N^p) \to T(M) \text{ along } g \cdot i.$$

Theorem B(p) is a local extension of the classical theorem that asserts the existence and uniqueness of a parallel unit vector field along a curve $g:N^1=(a, b) \to M$ in terms of initial data at a point $N^0 \in N^1$. In [1], a procedure is developed that proves Theorem B(p) and at the same time the classical theorem on the local existence and uniqueness of a C^ω minimal immersion in terms of initial data on a codimension one submanifold. The procedure makes use of certain differential forms on the p plane bundles over N and M. The solutions of both problems appear as integral manifolds that pass through the initial data. Their existence and uniqueness is a consequence of the Cartan-Kahler Theorem. Here, we sketch a proof of Theorem B(p) using the Cauchy-Kowalewski Theorem directly.

Let $\bar{n} \in N^{p-1}$. The assumptions on N^{p-1} and H imply the existence of a coordinate neighborhood $(V, z_1, \cdots, z_p)$ of $\bar{n}$ in N^p where $\|\partial/\partial z_p\| \equiv 1$, $N^{p-1} \cap V$ is the slice $z_p = 0$ and the integral manifolds of the distribution $\perp H$ are the slices $z_i = \text{constant}$ $i = 1, \cdots, p-1$. Because of the compatibility condition on G^{p-1} and G^p it may also be assumed that there are fields of orthonormal frames

$\{Z_1, \cdots, Z_p = \partial/\partial z_p\}$ on V and $\{Y_1, \cdots, Y_m\}$ along $g|_V$ with the property that a C^ω T.B.I. G defined along $g|_V$ extends the initial data along $g|_V$ if and only if its matrix representation (r_{ki}) with respect to these frames $(G(Z_i) = \sum_k r_{ki}, Y_k\, i = 1, \cdots, p)$ satisfies the equations

$$r_{ki} = \delta^{ki}, \qquad k = 1, \cdots, m, \quad i = 1, \cdots, p-1,$$

$$r_{1p} = , \cdots, = r_{p-1p} = 0 \qquad \text{on } V$$

and

$$r_{pp} = 1, \qquad r_{p+1p} = , \cdots, = r_{mp} = 0 \qquad \text{on } V \cap N^{p-1}.$$

It follows that the T.B.I.'s G that extend the initial data on V are in bijective correspondence with the $m-p$ tuples $(r_{p+1p}, \cdots, r_{mp})$ of C^ω functions on V that vanish on $N^{p-1} \cap V$. The condition that G be parallel along $g|_V$ is expressed by the vanishing, for each $n \in V$, of the projection of $\sum_{i=1}^{p} \nabla_{Z_i(n)} G(Z_i)$ into $\perp G(N_n^p)$. On some, perhaps smaller, neighborhood of $\bar{n}$ this condition is equivalent to the Cauchy-Kowalewski system:

$$0 = \left\langle \sum_{i=1}^{p} \nabla_{Z_i(n)} G(Z_i),\ Y_j(n) \right\rangle$$

$$= \left\langle \sum_{i=1}^{p-1} \nabla_{Z_i(n)} Y_i,\ Y_j(n) \right\rangle + \sum_{l=p}^{m} r_{lp} \langle \nabla_{Z_p(n)} Y_l,\ Y_j(n) \rangle + \frac{\partial r_{jp}}{\partial z_p}(n),$$

$$j = p+1, \cdots, m.$$

Thus, on some sufficiently small neighborhood $V^{\bar{n}}$ of any point $\bar{n} \in N^{p-1}$, there is a unique C^ω parallel T.B.I. $G^{\bar{n}}$ that extends the initial data along $g|_{V^{\bar{n}}}$. A neighborhood U of N^{p-1} in N^p can then be constructed on which there is a unique C^ω parallel T.B.I. G that extends the initial data along $g|_U$ so that for each $\bar{n} \in N^{p-1}: G|_{U \cap V^{\bar{n}}} = G^{\bar{n}}|_{U \cap V^{\bar{n}}}$.

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