DIMENSION AND MULTIPLICITY FOR GRADED ALGEBRAS[1]

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We want to reconsider a problem that goes back to Hilbert [3]. Let $R=\sum R^p$ be a commutative algebra which is graded by the nonnegative integers and finitely generated over $R^0=F$, which for simplicity is a field. Let $M=\sum M^p$ be a finitely generated graded R-module, with p again restricted to the nonnegative integers. Each component M^p is a finite-dimensional vector space over F. If R is generated over F by elements homogeneous of degree one then Hilbert proved that there is a polynomial

$$H_M(p) = e(M)p^{n-1}/(n-1)! + \cdots$$

such that $H_M(p) = \dim M^p$ for p large. With the understanding that the zero polynomial is of degree -1, we may call n the *dimension* of M. The coefficient $e(M)$ is a nonnegative integer, the *multiplicity* of M.

Unfortunately, if R is not generated by elements of degree one, it is not usually true that $\dim M^p$ is eventually given by a polynomial in p. (For example, let $M=R=F[x]$ where x is an indeterminant of degree two.) The more general case, where the generators of R are of degree greater than one, arises naturally. We need a substitute for the Hilbert polynomial and it turns out that the Poincaré series

$$P(M) = \sum (\dim M^p)t^p$$

of the module is a good substitute. In the classical situation the relation between H_M and $P(M)$ is such that H_M is of degree at most $n-1$ if and only if $(1-t)^nP(M)$ is a polynomial in t. Moreover, if H_M is of degree exactly $n-1$ then $e(M)$ is the value of $(1-t)^nP(M)$ for $t=1$. We intend to show how these facts generalize. The details of the proofs will be given elsewhere.

In [4] Serre gave a homological treatment of dimension and multiplicity for local rings. Following Serre, we wish to define the multi-

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plicity of a graded module M as an Euler characteristic of the complex

$$\mathrm{Tor}^R(F, M) = \sum \mathrm{Tor}_i^R(F, M).$$

Let $C(R)$ be the category of all finitely generated graded modules over R, and all homomorphisms which are homogeneous of degree zero. Each $\mathrm{Tor}_i^R(F, M)$ is a finite-dimensional graded vector space, a module of the category $C(F)$. As Fraser [2] has observed, it is natural to consider the Grothendieck groups $K(R)$ and $K(F)$ of the two categories, and attempt to define a multiplicity homomorphism $\chi_R: K(R) \rightarrow K(F)$. We set

$$\chi_R(M) = \sum (-1)^i [\mathrm{Tor}_i^R(F, M)]$$

where $[\mathrm{Tor}_i^R(F, M)]$ is the image in $K(F)$. This makes sense if $\mathrm{Tor}^R(F, M)$ is a finite complex. Surprisingly, the formula makes sense in the "completion" of $K(F)$ whether or not $\mathrm{Tor}^R(F, M)$ is finite. Since a graded vector space V is determined by the dimensions of its components, associating to V its Poincaré polynomial $P(V)$ identifies $K(F)$ with the polynomial ring $\boldsymbol{Z}[t]$ over the integers. Using Eilenberg's technique [1] of minimal resolutions it is easy to prove a lemma which insures that the above alternating sum is a well-defined formal power series in t.

LEMMA. *The pth component of* $\mathrm{Tor}_i^R(F, M)$ *is zero if* $p < i$.

From the long exact sequence for Tor we have a homomorphism $\chi_R: K(R) \rightarrow \boldsymbol{Z}[[t]]$ into the formal power series ring.

If every module in $C(R)$ has a finite resolution by free modules in $C(R)$, i.e., if $C(R)$ is of finite global dimension, then χ_R has values in the polynomial ring $\boldsymbol{Z}[t]$. In this case it is also true that $K(R)$ is a ring, with the product of two of the generators given by

$$[M][N] = \sum (-1)^i [\mathrm{Tor}_i^R(M, N)].$$

This formula always makes sense in case one of the modules is free. The free modules of $C(R)$ are all of the form $R \otimes_F V$ for V in $C(F)$. Thus in general $K(R)$ is a module over $K(F) = \boldsymbol{Z}[t]$.

THEOREM 1. *For any* R, $\chi_R: K(R) \rightarrow \boldsymbol{Z}[[t]]$ *is a homomorphism of* $\boldsymbol{Z}[t]$*-modules. If* $C(R)$ *has finite global dimension then* $\chi_R: K(R) \rightarrow \boldsymbol{Z}[t]$ *is a ring isomorphism.*

Associate to a graded finite-dimensional vector space its total dimension. This yields a ring homomorphism $\dim: \boldsymbol{Z}[t] \rightarrow \boldsymbol{Z}$ which is the natural augmentation, the function which assigns to a polynomial its value for $t=1$. If $C(R)$ is of finite global dimension then composing with χ_R gives a ring homomorphism $e_R: K(R) \to \boldsymbol{Z}$ and we have Serre's definition of the multiplicity in our situation:

$$e_R(M) = \sum (-1)^i \dim \operatorname{Tor}_i^R(F, M).$$

The category $C(R)$ is of finite global dimension if (and probably only if) R is a polynomial algebra $F[x_1, \cdots, x_n]$ generated by indeterminants which are homogeneous of positive degrees. In this case the Koszul complex can be used to compute multiplicities. Let $H_i(\mathbf{x}, M)$ be the ith homology module of the Koszul complex of $\mathbf{x}=(x_1, \cdots, x_n)$ and M.

THEOREM 2. *Let $R=F[x_1, \cdots, x_n]$ be a polynomial algebra generated by indeterminants of positive degrees $d_1, \cdots, d_n$. Then*

$$\chi_R(M) = \sum (-1)^i [H_i(\mathbf{x}, M)].$$

In particular, $\chi_R(F) = \prod (1-t^{d_i})$.

In the classical situation the indeterminants are all of degree one, so $\chi_R(F) = (1-t)^n$. This suggests the following theorem, which relates the multiplicity of a module to its Poincaré series.

THEOREM 3. *For any R and any M in $C(R)$, $\chi_R(M) = \chi_R(F)P(M)$.*

COROLLARY 1. *If $C(R)$ is of finite global dimension then $\chi_R(F)P(M)$ is a polynomial in t and $e_R(M)$ is the value of this polynomial for $t=1$.*

We can always reduce to the case of finite global dimension by regarding R as a quotient of a polynomial algebra S. An R-module M becomes an S-module. The Poincaré series is unaffected, and $\chi_S(M)$ and $\chi_S(F)$ are polynomials.

COROLLARY 2. *$\chi_R(M) = P(M)/P(R)$, and $P(M)$ and $\chi_R(M)$ are rational functions.*

The relation $\chi_R(M) = P(M)/P(R)$ follows from the fact that $\chi_R(F)P(R) = \chi_R(R) = 1$.

COROLLARY 3. *$\chi_R(M) = 0$ if and only if $M = (0)$.*

Call M of *dimension* at most n if there are positive integers $d_1, \cdots, d_n$ such that $P(M)\prod(1-t^{d_i})$ is a polynomial in t.

THEOREM 4. *The R-module M is of dimension at most n if and only if there exist homogeneous elements $y_1, \cdots, y_n$ in R such that M is finitely generated over the subalgebra $F[y_1, \cdots, y_n]$.*

If $P(M)\prod(1-t^{d_i})$ is a polynomial it is not true that we can always choose $y_1, \cdots, y_n$ of degrees $d_1, \cdots, d_n$. For example, let $M=R=F[x, y]$ where x is an indeterminant of degree two and y is a nonzero element of degree one with $y^2=0$. The Poincaré series is

$$P(M) = (1+t)/(1-t^2) = 1/(1-t)$$

but R contains no element y_1 of degree one with M finitely generated over $F[y_1]$.

COROLLARY. *If $R=F[y_1, \cdots, y_n]$ then every M in $C(R)$ is of dimension at most n.*

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