A SELF-UNIVERSAL CRUMPLED CUBE WHICH IS NOT UNIVERSAL

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C. E. Burgess and J. W. Cannon [2, §10] have asked whether each self-universal crumpled cube is universal. In this note we give a negative answer to their question by showing that the familiar solid Alexander horned sphere K is not universal. Casler has shown that K is self-universal [3].

A *crumpled cube* C is a space homeomorphic to the union of a 2-sphere S topologically embedded in the 3-sphere S^3 and one of its complementary domains. The *boundary of* C, denoted Bd C, is the image of S under the homeomorphism. A *sewing* h *of two crumpled cubes* C *and* C^* is a homeomorphism of Bd C to Bd C^*. The *space* $C\cup_h C^*$ *given by a sewing* h is the identification space obtained from the (disjoint) union of C and C^* by identifying each point p in Bd C with $h(p)$ in Bd C^*.

A crumpled cube C is *universal* if, for each crumpled cube C^* and each sewing h of C and C^*, the space $C\cup_h C^*$ is topologically equivalent to S^3. Similarly, a crumpled cube C is *self-universal* if $C\cup_f C=S^3$ for each sewing f of C to itself.

1. A bad sewing. In order to define the desired sewing of the solid Alexander horned sphere K to another crumpled cube K^*, we describe an upper semicontinuous decomposition of S^3 into points and almost tame arcs.

Let H_1 and H_2 denote the upper and lower half spaces of E^3, and P the xy-plane. Let A_0 denote a solid double torus embedded in E^3 as shown in Figure 1 such that A_0 intersects P in two disks D_1 and D_2. Letting T_1 and T_2 denote solid double tori embedded in A_0 as shown in Figure 1, we define A_1 as $T_1\cup T_2$. Assuming sets $A_0, A_1, \cdots, A_{n-1}$ have been defined, we let A_n be the union of 2^n solid double tori contained in A_{n-1} such that each double torus T of A_{n-1} contains exactly two components of A_n, which are embedded in T just as T_1 and T_2 are embedded in A_0.

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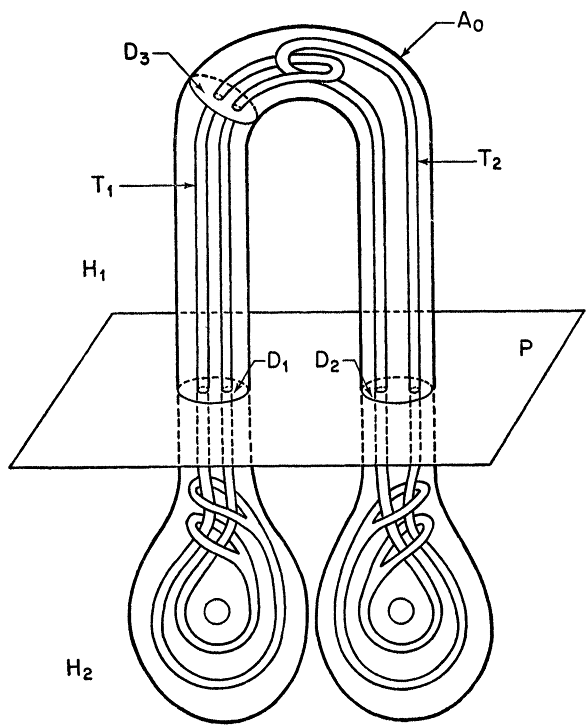

FIGURE 1

Let G' denote the upper semicontinuous decomposition of E^3 whose nondegenerate elements are the components of $\bigcap_{j=1}^{\infty} A_j$. By requiring that the components of A_n become skinny as n gets large, we force the nondegenerate elements of G' to be arcs which are locally tame except at their lower end points.

With the addition of an ideal point ∞, G' extends to a decomposition G of S^3.

THEOREM 1. *The decomposition space S^3/G is not homeomorphic to S^3.*

The proof of Theorem 1 is discussed in the next section.

THEOREM 2. *Let K denote the solid Alexander horned sphere. There exists a sewing h of K to a crumpled cube K^* such that $K \cup_h K^*$ is not homeomorphic to S^3.*

PROOF. Let π denote the natural projection of S^3 to S^3/G, and let $H_i^* = H_i \cup \{\infty\}$ $(i=1, 2)$. Note that $\pi(H_1^*)$ is topologically equivalent to K, and $\pi(H_2^*)$ is a crumpled cube K^*. The required sewing h is the one induced by π such that $K\cup_h K^*$ and S^3/G are homeomorphic.

REMARK. The procedure for defining K^* is suggested by Stallings' crumpled cube [4].

2. **Slicing homeomorphisms.** Let k be a nonnegative integer. A homeomorphism h of Bd $A_0\cup D_1\cup D_2\cup D_3$ into A_0 such that $h|$ Bd A_0 = identity is said to be *slicing at stage k* if, for each solid double torus T of A_k, each component of $T\cap h(D_i)$ $(i=1, 2, 3)$ is a disk embedded in T just like a component of $T\cap P$.

A homeomorphism h slicing at stage k is said to satisfy Property P_k if for some double torus T of A_k there exist components X_1, X_2, and X_3 of $T\cap h(\cup D_i)$ such that

(a) $X_1\cup X_2\subset h(D_{i_1}\cup D_{i_2})$,

(b) $X_3\cap h(D_{i_1}\cup D_{i_2})=\varnothing$,

(c) X_3 separates X_1 from X_2 in T.

Theorem 1 is an immediate consequence of [1, Theorem 2] and the following lemmas.

LEMMA 1. *If h is a homeomorphism slicing at stages k and k+1 and satisfying Property* P_k, *then h satisfies Property* P_{k+1}.

LEMMA 2. *If h is a homeomorphism slicing at stage k+1, then there exists a homeomorphism h* slicing at stages k and k+1 such that for each component T of* A_{k+1}, $T\cap h(D_i)=\varnothing$ *implies* $T\cap h^*(D_i)=\varnothing$ $(i=1,2,3)$.

LEMMA 3. *Every homeomorphism slicing at stage k satisfies Property* P_k.

LEMMA 4. *If there exists a nonnegative integer k and a homeomorphism g of* Bd $A_0\cup D_1\cup D_2\cup D_3$ *into* A_0 *such that* $g|$ Bd A_0 = *identity and each component T of* A_k *intersects at most one of the disks* $g(D_i)$, *then there exists a homeomorphism h slicing at stage k that fails to satisfy Property* P_k.

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