GENERALIZED THOM SPECTRA AND TRANSVERSALITY FOR SPHERICAL FIBRATIONS[1]

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1. A Poincaré duality space (abbreviated P.D. space) of dimension $n \geqq 0$ is a finite complex M with the following property.

Let M be embedded in S^{n+k}, k large, and let R be a regular neighborhood; then the inclusion $\partial R \subseteq R$, when converted to a fibration, has fiber a $(k-1)$-sphere.

Similarly a Poincaré cobordism $(W; M_0, M_1)$ of dimension $n+1$ is a triad with the following property.

Let $W; M_0, M_1$ be embedded in $S^{n+k} \times (I; \{0\}, \{1\})$ with relative regular neighborhood R (i.e. $R \cap S^{n+k} \times \{i\} = Q_i$ is a regular neighborhood of M_i in $S^{n+k} \times \{i\}, i=0, 1$). Let $\bar{\partial} R = \text{closure } \partial R - S^{n+k} \times \{0, 1\}$. Then $\bar{\partial} R \subseteq R$ is a $(k-1)$-spherical fibration and $\bar{\partial} R \cap Q_i = \partial Q_i \subseteq Q_i$ is the induced $(k-1)$-spherical fibration.

A P.D. pair $M, \partial M$ is a P.D. cobordism $M; \partial M, \emptyset$. If $W; M_0, M_1$ is a P.D. cobordism then M_0, M_1 are P.D. spaces of one lower dimension. For a P.D. space M let $\nu_k(M)$ be the fibration corresponding to $\partial R \subseteq R$; for a P.D. cobordism $W; M_0, M_1$ let $\nu_k(W; M_0, M_1)$ be the fibration corresponding to $\bar{\partial} R \subseteq R$.

A Generalized Thom Spectrum is a spectrum defined as follows: let $\xi_k : E_k \to B_k$ be a sequence of $(k-1)$-spherical fibrations, $k \geqq 1$. Let $\psi_k : B_k \to B_{k+1}$ be maps covered by spherical-fibration maps $\phi_k : \xi_k \oplus \epsilon \to \xi_{k+1}$.

Let the Thom complex $T(\xi^{x^i})$ be the space $\mathfrak{M}_{\xi_k} \cup cE_k$, i.e. the mapping cylinder of $\xi_k : E_k \to B_k$ union the cone on E_k with the top of the mapping cylinder identified with the base of the cone. There are natural maps $\sum T(\xi_k) \to T(\xi_{k+1})$. This forms the generalized Thom spectrum T.

Let S be the spectrum got by taking $B_k = B_{k+1} \cdots = \text{point}$; thus S is the sphere spectrum. If T is any spectrum as above, we assume that there are base points in each B_k, preserved by ψ. This gives an inclusion of spectra $S \subseteq T$.

A T-P.D. space (or simply T-space) is a P.D. space M together with maps of spherical fibrations $f_k : \nu_k(M) \to \xi_k$ so that

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$$\begin{array}{ccc} \nu_k \oplus \epsilon & \xrightarrow{\cong} & \nu_{k+1} \\ \downarrow f_k \oplus 1 & & \downarrow f_{k+1} \\ \xi_k \oplus 1 & \xrightarrow{\phi_k} & \xi_{k+1} \end{array}$$

commutes (k large). We denote such an object by (M, f).

Similarly, a T-cobordism is a P.D. cobordism W; M_0, M_1 together with maps $g_k : \nu(W;\ M_0,\ M_1) \to \xi_k$. Two T-spaces $(M_0,\ f^0)$, $(M_1,\ f^1)$ are said to be T-cobordant iff there is a T-cobordism W; M_0, M_1; g so that $f_k = g_k \circ i_k^j$, where i_k^j is the natural map $\nu_k(M_j) \subseteq \nu_k(W;\ M_0,\ M_1)$, $j=0,\ 1$.

T-cobordism is an equivalence relation, and the operation of taking disjoint union defines a group on equivalence classes, so that we get a graded group Ω_*^T (graded by dimension). Of course, if we were dealing with PL or orthogonal bundles we would have the formula $\Omega_*^T = \pi_* T$, where π_* denotes stable homotopy of the spectrum. However, it is well known that this is not the case for generalized Thom spectra. In particular, if T arises from ξ_k = universal orientable spherical fibration, i.e. $B_k = \mathrm{BSG}(k)$, then it is known that $\pi_* T$ is finite whereas Ω_{4i}^T is of infinite order because there is a well-defined nontrivial homomorphism (the index) to the integers.

Our purpose then is to study the relationship between Ω_*^T and $\pi_* T$. In this paper, we state some important results. A full exposition will appear later.

2. The Pontrjagin-Thom construction yields a homomorphism $p : \Omega_*^T \to \pi_* T$. We will attempt to elucidate some properties of p. We restrict our attention to the case where ξ_k is oriented and ϕ_k respects orientation.

We will first need to analyze a certain graded group Q_*, which is defined as the kernel of $p : \Omega_*^S \to \pi_* S$. It is easily seen that there are short exact sequences

$$0 \to Q_i \to \Omega_i^S \xrightarrow{p} \pi_i S \to 0.$$

(The fact that p is onto follows immediately from the fact that the isomorphism between ordinary framed cobordism of manifolds and $\pi_* S$ factors through Ω_*^S.)

LEMMA 1. *For* $i \geqq 5$, $Q_i = \pi_i\ G/\mathrm{Top}$. *Recall* $\pi_i\ G/\mathrm{Top} = 0$, i *odd*; Z, $i \equiv 0\ (4)$; Z_2, $i \equiv 2\ (4)$.

This is really a paraphrase of the fact that two elements in Q_i, $i \geqq 5$, are equal if and only if they have the same index-Arf invariant.

Recent work of F. Quinn suggests that this is true for $i=3, 4$ as well. We conjecture that this is true, in fact, for all i. (For the definition of the Arf invariant on Ω^S_{4i+2}, see [1].)

3. Now let N be a compact PL manifold of dimension r, and let $\xi: E \to B$ be a $(k-1)$-spherical fibration $T(\xi) = \mathfrak{M}_\xi \cup cE$. Let $f: N \to T(\xi)$. Set $R = f^{-1}(\mathfrak{M}_\xi)$, $Q = f^{-1}(cE)$, $D = f^{-1}(E)$, $(E = \mathfrak{M}_\xi \cap cE)$ so that $D = R \cap Q$. We say that f is transverse regular iff R, Q are codimension 0 submanifolds of N, $\partial R = \partial Q = D$ and

$$\begin{array}{ccc} D & \longrightarrow & E \\ |\subseteq & & |\subseteq \\ R & \longrightarrow & \mathfrak{M}_\xi \end{array}$$

is, up to homotopy, a map of $(k-1)$-spherical fibrations. Thus, R is a P.D. space of dimension $r-k$. A similar definition can be made if N is a manifold with boundary.

THEOREM 1. *Let N, ξ be as above, with $k \geqq 6$, $r \leqq 3k-3$, or with $k \geqq 3$, $r \leqq 2k-1$. Let $f: N \to T(\xi)$ be a continuous map. Then there are obstructions to deforming f to a transverse regular map which lie in $H^i(N, Q_{i-k-1})$.*

Actually, for a suitable definition of transverse regularity, Theorem 1 holds true for N an arbitrary complex.

Theorem 1 can be elucidated by considering the following construction:

First, $T(\xi) - *$ is an open set $\mathcal{A}$ of $T(\xi)$. (Here, $*$ is the cone point of cE.) We may assume that there is a contractible open set $\mathcal{O} \subseteq \mathcal{B}$; $\mathfrak{M}_\xi|_\mathcal{O} \subseteq \mathfrak{M}_\xi$, and $\mathfrak{M}_\xi|_\mathcal{O} \cup (cE - E)$ is an open set $\mathcal{B}$ of $T(\xi)$. $\{\mathcal{A}, \mathcal{B}\}$ is an open cover of $T(\xi)$. Let $\bar{\omega}(\xi)$ be the semisimplicial complex whose j-simplices are maps $\sigma: \Delta^j \to T(\xi)$ where $\sigma(\Delta^j) \subseteq \mathcal{A}$ or $\sigma(\Delta^j) \subseteq \mathcal{B}$ and where σ is transverse regular on Δ^j and all its faces. There is a natural map $\bar{\omega}(\xi) \to T(\xi)$. The obstructions of Theorem 1 measure the possibility that a map to $T(\xi)$ may be deformed so that it can be lifted to $\bar{\omega}(\xi)$. If T is a Generalized Thom Spectrum defined by ξ_k, $k=1, 2, \cdots$ then we can make $\{\bar{\omega}(\xi_k)\}$ into a spectrum $\overline{\mathcal{W}}(T)$ so that the maps $\bar{\omega}(\xi_k) \to T(\xi_k)$ define a map of spectra $\overline{\mathcal{W}}(T) \to T$.

We can define a semisimplicial complex $\omega(\xi)$ similar to $\bar{\omega}(\xi)$ by simply ignoring the condition that image $\sigma \subseteq \mathcal{A}$ or $\mathcal{B}$, and requiring only that σ be transverse regular on Δ^j and all its faces. Thus $\bar{\omega}(\xi) \subseteq \omega(\xi)$. Similarly, given a Generalized Thom Spectrum, the complexes $\omega(\xi_k)$ form a spectrum $\mathcal{W}(T)$. It turns out that $\pi_* \mathcal{W}(T) = \Omega^T_*$, and that the natural map $\mathcal{W}(T) \to T$ is a representative, on the level of spectra, of the Pontrjagin-Thom map.

THEOREM 2. *There is an exact sequence*

$$\cdots \to Q_i \to \pi_i\overline{\mathcal{W}}(T) \to \pi_i(T) \to Q_{i-1} \to \cdots.$$

This follows from Theorem 1.

We conjecture that $\overline{\mathcal{W}}(T) \subseteq \mathcal{W}(T)$ is a homotopy equivalence. This would allow the substitution of Ω_i^T for $\pi_i\overline{\mathcal{W}}(T)$ in the exact sequence above.

4. Consider a Generalized Thom Spectrum T. If (M, f) is a T-space and if $g: M' \to M$ is a degree-one map covered by $\psi_k: \nu_k(M') \to \nu_k(M)$, large k, then setting $f'_k = f_k \circ \psi_k$ defines another T-space (M', f'). We say that (M', f') is related to (M', f'). Consider the equivalence relation $\cong$ generated by this relation; dividing by $\cong$ defines a quotient group T_*^T of Ω_*^T, and it is easy to see that $p: \Omega_*^T \to \pi_* T$ factors through $p_0: T_*^T \to \pi_* T$.

THEOREM 3. *p_0 is a monomorphism; in fact, for $i \geqq 6$, $\neq 4k+3$, p_0 is an isomorphism; for $i \geqq 7$, $=4k+3$,* coker *p_0 is at most Z_2.*

Theorem 3 follows from Theorem 1 and some of the techniques of Theorem 2. We outline the proof: To show p_0 is monic, we first show that if M, f represents an element in the kernel of $p_0: T_i^T \to \pi_i T$ then there is an M^1, f^1 related to M, f so that $[M^1, f^1] \in \Omega_i^T$ is in the image of $\pi_i\overline{\mathcal{W}}(T) \to \pi_i\mathcal{W}(T) = \Omega_i^T$. One then shows by means of the exact sequence that M^1, f^1 is T cobordant to M^2, f^2 so that M^2, f^2 is actually a G-framed P.D., space, i.e. M^2, f^2 is an S-space, and represents an element in Q_i. It is easy to show then that there is a T-space (in fact, an S-space) M^3, f^3 related to M^2, f^2, and M^3, f^3 is T-cobordant (in fact, S-cobordant) to zero. Thus M, f represents zero in T_i^T, and p_0 is monic.

To show the rest of Theorem 3, we merely use the computation of Q_* of Lemma 1.

5. Some conjectures are suggested by the above. First, let O^i denote ith loop space (instead of the customary notation Ω^i). Then given the spectrum T, $\mathcal{W}(T)$, we get spaces $A(T) = \lim\, O^iT(\xi_i)$, $B(T) = \lim\, O^i\omega(\xi_i)$ so that $\pi_*A(T) = \pi_*T$, $\pi_*B(T) = \pi_*\mathcal{W}(T) = \Omega_*^T$. The map of spectra $\mathcal{W}(T) \to T$ becomes a map $B(T) \to A(T)$; and if we have a map of Generalized Thom Spectra $T \to U$ we get a homotopy commutative square

$$\begin{array}{ccc} B(T) & \to & B(U) \\ \downarrow & & \downarrow \\ A(T) & \to & A(U). \end{array}$$

Let $C(T) = \text{fiber } B(T) \to A(T)$. Then we conjecture that $C(T) = G/\text{Top}$ and

$$\begin{array}{ccc} G/\text{Top} & = & G/\text{Top} \\ \downarrow & & \downarrow \\ B(T) & \to & B(U) \\ \downarrow & & \downarrow \\ A(T) & \to & A(U) \end{array}$$

is a map of fibrations. If we abandon the condition that T be defined by *orientable* fibrations S_k, we then conjecture that $C(T)$ is a space whose homotopy realizes the L-groups of some appropriate group (see [3], [4]).

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