ON PASTING BALLS TO HANDLEBODIES

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Throughout this paper all spaces will be simplicial complexes and all maps will be piecewise linear. We shall denote the boundary, closure, and interior of a space X by $\mathrm{bd}(X)$, $\mathrm{cl}(X)$ and $\mathrm{int}(X)$ respectively. Let X be a space and Y a connected subspace. Then we shall denote the natural map induced by inclusion from $\pi_1(Y)$ into $\pi_1(X)$ by $\pi_1(Y)\rightarrow\pi_1(X)$.

We shall say that a submanifold X of a manifold Y is *properly embedded* in Y if $X\cap\mathrm{bd}(Y)=\mathrm{bd}(X)$. A *handlebody* is a 3-manifold homeomorphic to the regular neighborhood of a compact 1-complex embedded in E^3. If T_n is a handlebody and l is a simple loop in $\mathrm{bd}(T_n)$, we can attach a disk to T_n by identifying the boundary of the disk with l. We may attach a thickened disk or a ball in a similar way to T_n and obtain a 3-manifold. When we perform the operation above we shall say that we have *pasted a ball to T_n along l.* We shall denote the smallest normal subgroup of $\pi_1(T_n)$ containing $[l]$ by $N(l)$.

It is the purpose of this article to prove:

THEOREM. *Let T_n be a handlebody of genus n. Let l be a simple loop in* $\mathrm{bd}(T_n)$ *such that $\pi_1(T_n)/N(l)$ is free on $n-1$ generators. Then the 3-manifold obtained by pasting a ball to T_n along l is a handlebody of genus $n-1$.*

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PROOF. It follows from a theorem of Whitehead (see [2, p. 167, Theorem N3]) that $[l]$ can be taken to be a generator of $\pi_1(T_n)$. Let T_n' be homeomorphic to T_n under a map $h: T_n\rightarrow T_n'$. Then we can paste T_n to T_n' along regular neighborhoods in $\mathrm{bd}(T_n)$, $\mathrm{bd}(T_n')$ of l and $h(l)$ respectively, to obtain a 3-manifold M.

It is a consequence of Van Kampen's Theorem that $\pi_1(M)$ is free on $2n-1$ generators. Now $\pi_1(\mathrm{bd}(M))$ is not free, so $\pi_1(\mathrm{bd}(M))\rightarrow\pi_1(M)$ is not one-one. It follows from the loop theorem [3] that there is a disk properly embedded in M such that $\mathrm{bd}(\mathfrak{D})$ is not

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homotopic to a point in bd(M). We pick $\mathfrak{D}$ so that bd$(\mathfrak{D})\cap$bd$(T_n)\cap$bd(T_n') is a finite set of minimal cardinality. (In particular, bd$(\mathfrak{D})$ crosses bd$(T_n)\cap$bd(T_n') at every point in the above set.) By general position cl$(\mathfrak{D}\cap($bd$(T_n)\cap$int$(M)))$ may be taken to be a family of simple arcs $\alpha_1, \cdots, \alpha_m$ whose interiors are pairwise disjoint. We may assume further that each of the above arcs meets l in at most one point and that said arc crosses l at this point. We may choose a component of $\mathfrak{D}-\bigcup_{i=1}^m \alpha_i$ whose closure contains at most one of the α_i. The closure of this component is a disk $\mathfrak{D}_1$ which we may assume is properly embedded in T_n.

We claim that we may assume bd$(\mathfrak{D}_1)$ is not nullhomotopic in bd(T_n). This can be seen as follows: If bd$(\mathfrak{D}_1)$ is nullhomotopic on bd(T_n), it bounds a subdisk $\mathfrak{D}_2$ of bd(T_n). Since l is not nullhomotopic on bd(T_n) and since l meets bd$(\mathfrak{D}_2)$ at most once, bd$(\mathfrak{D}_2)$ does not meet l. It follows that both endpoints of the arc bd$(\mathfrak{D}_2)\cap$bd(M) lie on a single loop l_1 of the boundary of a regular neighborhood of l in bd(T_n). Now $l_1\cap\mathfrak{D}_2$ is an arc which separates $\mathfrak{D}_2$ into two disks. One of these disks lies on bd(M). It is now easily seen that bd$(\mathfrak{D})$ did not meet bd$(T_n)\cap$bd(T_n') in a set of minimal cardinality. The claim follows.

Thus bd$(\mathfrak{D}_1)$ may be assumed to be nonnullhomotopic on bd(T_n). It follows from our assumptions above that bd$(\mathfrak{D}_1)\cap l$ is at most a single point. We will show that this intersection may be taken to be a single point.

Assume $\mathfrak{D}_1$ does not meet l, i.e., $\mathfrak{D}_1=\mathfrak{D}$.

We remove a regular neighborhood of $\mathfrak{D}_1\cup h(\mathfrak{D}_1)$ from M. Denote the closure of the resulting manifold by M_1. Now bd(M_1) is a closed surface in M. Since $\pi_1(M)$ is free, $\pi_1($bd$(M_1))\to\pi_1(M)$ is not an injection whenever genus bd$(M_1)>0$. Let λ be a loop on bd(M_1) which is nullhomotopic in M. Now $M_1\cap$cl$(M-M_1)$ is a collection of disks. It follows from Lemma 1.2 in [1] that λ is nullhomotopic in M_1. It follows from the loop theorem in [3] that there exists a disk E properly embedded in M_1 such that bd(E) is not nullhomotopic in bd(M_1). Clearly we may choose bd(E) to lie on bd$(M_1)\cap$bd(M) since bd$(M_1)-$bd(M) is a collection of disks. By the arguments above either we can find a sequence of compact 3-submanifolds M_i, $i=1, \cdots, m$, such that

(1) $M_i\supset M_i+1$ for $i=1, \cdots, m-1$;

(2) $l\subset$bd(M_i) for $i=1, \cdots, m$;

(3) genus bd$(M_i)>$genus bd(M_{i+1}) for $i=1, \cdots, m-1$;

(4) bd$(M_{i+1})-$bd(M_i) is the union of four open disks, for $i=1, \cdots, m-1$, and $m=$genus bd$(M)/2$; or there is a disk $\overline{\mathfrak{D}}$

properly embedded in T_n such that

(1) $\mathrm{bd}(\bar{\mathfrak{D}})$ is not nullhomotopic in $\mathrm{bd}(T_n)$;

(2) $\mathrm{bd}(\bar{\mathfrak{D}})\cap l$ is a single point at which $\mathrm{bd}(\bar{\mathfrak{D}})$ crosses l.

The former is impossible since the genus of $\mathrm{bd}(M)$ is finite and since l must be nonnullhomotopic on $\mathrm{bd}(M_i)$ for each i by 1.2 in [**1**].

It follows that we can find a disk $\mathfrak{D}$ properly embedded in T_n such that $\mathrm{bd}(\mathfrak{D})$ is not nullhomotopic in $\mathrm{bd}(T_n)$ and $\mathfrak{D}\cap l$ is a single point at which $\mathrm{bd}(\mathfrak{D})$ crosses l.

Let H be a regular neighborhood of $\mathfrak{D}\cup l$. Now $\mathrm{cl}(\mathrm{bd}(H)\cap\mathrm{int}(T_n))$ is a disk E separating H from $\mathrm{cl}(T_n-H)$. By Van Kampen's Theorem

$$\pi_1(T_n) = \pi_1(H) \underset{\pi_1(E)}{*} \pi_1(\mathrm{cl}(T_n - H))$$

$$\prod_{i=1}^{n} Z = Z * G$$

where $G=\pi_1(\mathrm{cl}(T_n-H))$. It follows that $G=\prod_{i=1}^{n-1} Z$.

Using the method of proof above one can find a sequence $B_1, \cdots, B_m$ of 3-submanifolds of $\mathrm{cl}(T_n-H)$ such that

(1) $B_i\supset B_{i+1}$;

(2) $\mathrm{bd}(B_{i+1})-\mathrm{bd}(B_i)$ is the union of two open disks;

(3) $\mathrm{bd}(B_i)\supset E$;

(4) $\mathrm{cl}(\mathrm{cl}(T_n-H)-B_i)$ is a collection of i 3-balls for all i;

(5) the total genus of $\mathrm{bd}(B_m)$ is zero. It follows that $\mathrm{cl}(T_n-H)$ is a handlebody of genus $n-1$.

The desired result now follows by choosing an appropriate embedding of T_n in E^3 and pasting a ball to T_n across l in E^3.

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