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MAXIMAL MONOTONE OPERATORS AND NONLINEAR INTEGRAL EQUATIONS OF HAMMERSTEIN TYPE

BY FELIX E. BROWDER, DJAIRO G. DE FIGUEIREDO AND CHAITAN P. GUPTA

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A nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_G K(x, y)f(y, u(y))dy = w(x) \tag{1}$$

where G is a measure space with a σ-finite measure dy, the given function $w(x)$ and the unknown function $u(x)$ are defined on G. In operator-theoretic terms, the problem of determining the solutions of equation (1), with u, w lying in a given Banach space of functions on G, can be put in the form of a nonlinear functional equation

$$u + AN(u) = w \tag{2}$$

with the linear and nonlinear mappings A and N respectively given by

$$Av(x) = \int_G K(x, y)v(y)dy, \quad Nu(x) = f(x, u(x)). \tag{3}$$

In the present note, we apply the theory of maximal monotone operators in Banach spaces to establish general results on the existence of solutions of equation (2) for the reflexive Banach space X. Our results generalize the results of Browder-Gupta [8], Amann [1],

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Kolomý [11], Brezis [2], Kolodner [10], Dolph-Minty [9] and Vaĭnberg [14], [15].

We employ the following definitions: If X is a real Banach space, X^* its conjugate space, we let (w, u) denote the duality pairing between the element w in X^* and the element u in X. A subset G of the Cartesian product $X \times X^*$ is said to be *monotone* if $(v_1 - v_2, u_1 - u_2) \geqq 0$ for $[u_1, v_1] \in G$ and $[u_2, v_2] \in G$. A monotone set G is said to be *maximal monotone* if it is not properly contained in any other monotone set. Let T be a mapping of X into the set 2^{X^*} of all subsets of X^*.

The effective domain $D(T)$ of T is the subset of X given by $D(T) = \{u \mid u \in X,\ T(u) \neq \emptyset\}$. The range $R(T)$ of T is the subset of X^* given by $R(T) = \cup\{T(u) \mid u \in X\}$. The graph $G(T)$ of T is the subset of the Cartesian product $X \times X^*$ given by $G(T) = \{[u, v] \mid u \in X, v \in T(u)\}$.

T is said to be *monotone* if the graph $G(T)$ of T is a monotone set. T is said to be *maximal monotone* if the graph $G(T)$ of T is a maximal monotone set. The graph of the inverse mapping T^{-1} of X^* into 2^X is the subset of the Cartesian product $X^* \times X$ given by $G(T^{-1}) = \{[v, u] \mid [u, v] \in G(T)\}$.

It is immediate that a mapping T of a reflexive Banach space X into 2^{X^*} is maximal monotone if and only if the inverse mapping T^{-1} of X^* into 2^X is maximal monotone. T is said to be *pseudomonotone* if for any sequence $\{u_j\}$ in X which converges weakly to an element u_0 in X with

$$\limsup (T(u_j), u_j - u_0) \leqq 0$$

we have

$$\liminf (T(u_j), u_j - v) \geqq (Tu_0, u_0 - v) \quad \text{for all } v \in X.$$

Let N be a mapping of X into X^*. N is said to be *hemicontinuous* if it is continuous from each line segment in X to the weak topology in X^*. N is said to be *coercive* if $(N(v), v)/\|v\| \to \infty$ as $\|v\| \to \infty$.

THEOREM 1. *Let X be a reflexive Banach space, X^* its conjugate space. Let A be a maximal monotone mapping of X^* into 2^X and N a coercive hemicontinuous mapping of X into X^* which is monotone or bounded pseudomonotone.*

Then the range $R(I + AN)$ of the mapping $I + AN$ is all of X.

PROOF. For given w in X, we seek v in X with $N(v)$ in the domain $D(A)$ of A and

$$w \in v + AN(v). \tag{4}$$

The relation (4) is equivalent to

$$0 \in N(v) - A^{-1}(w - v). \tag{5}$$

Let T_w be the mapping of X into 2^{X^*} given by $T_w(v) = -A^{-1}(w-v)$ for v in X. Since $G(T_w) = -G(A^{-1}) + [w, 0]$, it follows that T_w is a maximal monotone mapping of X into 2^{X^*}. By hypothesis, we know in addition that N is a coercive hemicontinuous mapping of X into X^* with N monotone or bounded pseudomonotone. Hence it follows from results of Browder (Theorem 1, [5]) and Rockafellar [12], (see also Brezis-Crandall-Pazy [4]) that the range $R(T_w+N)$ is the whole of X^*.

Thus there exists a v_0 in X such that $0 \in T_w(v_0) + N(v_0) = N(v_0) - A^{-1}(w - v_0)$. Such an element v_0 is a solution of the relation (5) and hence of the relation (4). q.e.d.

COROLLARY 1. *Let X be a reflexive Banach space, A a hemicontinuous monotone mapping of X^* into X, N a coercive hemicontinuous monotone mapping of X into X^*.*

Then for every w in X, the equation $v + AN(v) = w$ has a solution v in X.

It suffices to note that the mapping A described in Corollary 1 is a maximal monotone mapping from X^* to X.

COROLLARY 2. *Let X be a reflexive Banach space and A a densely defined closed linear monotone mapping of X^* into X. Suppose that A^* is also a monotone mapping from X^* to X (this will be the case if in particular A is bounded).*

Let N be a coercive hemicontinuous monotone mapping of X into X^*.

Then for each w in X, there exists a v in X with $N(v)$ lying in $D(A)$ such that $v + AN(v) = w$.

By a theorem of Brezis [3], each closed densely defined monotone linear mapping A of X^* into X with A^* monotone is maximal monotone.

THEOREM 2. *Let X be a reflexive Banach space. Let A be a bounded linear monotone mapping of X^* into X and N be a hemicontinuous monotone mapping of X into X^*. Suppose that there is a nondecreasing nonnegative function ϕ defined on the set R^+ of nonnegative real numbers such that if for any u in X there is a v in X satisfying*

$$v + AN(v) = u$$

then $\|v\| \leq \phi(\|u\|)$.

Then the range $R(I+AN)$ of the mapping $I+AN$ is the whole of X.

PROOF. Let w be an arbitrary element of X. As in the proof of Theorem 1, it suffices to show that $0\in R(T_w+N)$ where T_w is the mapping of X into 2^{X^*} defined by

$$T_w(v) = - A^{-1}(w - v)$$

for v in X. Since A is a bounded linear monotone mapping of X^* into X, A is a maximal monotone mapping. Hence, as in the proof of Theorem 1, the mapping T_w of X into 2^{X^*} is maximal monotone. We first observe that the mapping T_w+N of X into 2^{X^*} is maximal monotone by the results of Browder [5] and Rockafellar [13].

Now for v in X and u in $(T_w+N)(v)$ we have that $u\in -A^{-1}(w-v)+N(v)$ i.e. $N(v)-u\in A^{-1}(w-v)$. This implies that $AN(v)-A(u)=w-v$ i.e. $v+AN(v)=w+A(u)$.

It now follows from our hypothesis that

$$\|v\| \leqq \phi(\|w + A(u)\|) \leqq \phi(\|w\| + \|A\|\,\|u\|).$$

Since ϕ is a nonnegative, nondecreasing function, it follows from the above inequality that the mapping $(T_w+N)^{-1}$ of X^* into 2^X is bounded. It then follows from the results of Browder [5] and Rockafellar [12] that the range $R(T_w+N)$ is whole of X^*. In particular, $0\in R(T_w+N)$. So there is v in X such that $0\in -A^{-1}(w-v)+N(v)$. This gives that $v+ANv=w$ and hence the theorem. q.e.d.

THEOREM 3. *Let X be a reflexive Banach space, A a bounded pseudo-monotone mapping of X^* into X, N a monotone hemicontinuous mapping of X into X^*. Suppose $k(r)$ and $c(r)$ are real valued functions of r in R^+ such that $k(r)+c(r)\to\infty$ as $r\to\infty$, that the following two inequalities are satisfied*:

(1) $(w, Aw) \geqq k(\|w\|)\|w\|, \qquad (w \in X^*)$

(2) $(w, u) \geqq c(\|w\|)\|w\|, \qquad (w \in X^*)$ *for all* $u \in N^{-1}(w)$.

Then $R(I+AN)=X$.

PROOF. $(I+AN)N^{-1}=N^{-1}+A$. Hence it suffices to prove that $R(N^{-1}+A)=X$. Since N is monotone and hemicontinuous, N^{-1} is maximal monotone from X^* to 2^X. Since A is bounded and pseudo-monotone, $A+N^{-1}$ will be onto X if $A+N^{-1}$ is coercive (Browder [7]). For w in X^* and u in $N^{-1}(w)$ we have

$$(*) \qquad (w, u + A(w)) \geqq [c(\|w\|) + k(\|w\|)]\cdot\|w\|.$$

This shows that $A+N^{-1}$ is coercive. Hence the theorem. q.e.d.

REMARK. We note that the inequality (*) will hold if condition (1) of Theorem 3 is replaced by the following:

$$(1') \qquad \|A(w)\| \leqq k_1(\|w\|), \qquad (w \in X^*)$$

where $k_1(r)$ is a real valued function of r in R^+ and $c(r) - k_1(r) \to \infty$ as $r \to \infty$.

COROLLARY 3. *Let X be a reflexive Banach space. A a bounded hemicontinuous monotone mapping of X^* into X, C a completely continuous mapping of X^* into X and N a hemicontinuous monotone mapping of X onto X^*. Suppose* (1) *and* (2) *hold with A replaced by $A + C$.*
Then $R(I + (A + C)N) = X$.

Corollary 3 follows from Theorem 3 immediately since $A + C$ is a bounded pseudomonotone mapping of X^* into X (Browder [7]).

We remark that the result of Corollary 3 is analogous to the results of Vaĭnberg for the equation (2) when A is a quasi-negative linear mapping of X^* into X (Vaĭnberg [15], [14]).

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UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637

UNIVERSITY OF ILLINOIS, CHICAGO, ILLINOIS 60680

NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115