TRIANGULATED INFINITE-DIMENSIONAL MANIFOLDS

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In this paper we extend almost all the results on infinite-dimensional Fréchet manifolds to apply to manifolds modeled on some l_2^f ($=\{x\in l_2|$ at most finitely many of the coordinates of x are nonzero$\}$) and we show (Theorem 14) that each l_2^f-manifold has a unique completion to an l_2-manifold. (We use l_2 to stand for the Hilbert space of all square-summable sequences of some infinite cardinality $\mathfrak{A}$. When we wish to be more specific, we write $l_2(\mathfrak{A})$.) Examples of $l_2^f(\aleph_0)$-manifolds include the metric S^∞ (the unit sphere in $l_2^f(\aleph_0)$) and the metric RP^∞, which may be regarded as the orbit space of S^∞ acted upon by the antipodal map. (The identity map from S^∞ or RP^∞ with the weak topology to the metric topology is a homotopy equivalence. (see [2]).) See also Theorem 16. In fact, we show (Theorems 15 and 17) that each l_2^f-manifold is a metric (see [2] for definition) simplicial complex and that each $l_2^f(\aleph_0)$-manifold is the 'metric' direct limit of finite-dimensional, closed, orientable manifolds. We conjecture that *the metric geometric realization of each connected singular s.s. complex (or Kan s.s. complex) is an $l_2^f(\mathfrak{A})$-manifold, for some cardinal $\mathfrak{A}$.* Most of the results here have been independently proved for the case of $l_2^f(\aleph_0)$ by T. A. Chapman [1] who used different methods. All manifolds in this paper are assumed paracompact.

DEFINITIONS. (1) If F is a TVS, define F^ω to be the countably infinite product and $F_f^\omega=\{\{x_i\}\in F^\omega|$ for at most finitely many i, $x_i\neq 0\}$.

(2) Let X and Y be spaces, $\mathfrak{U}$ be an open cover of Y, and $\mathfrak{F}$ a set of functions from X into Y. Then $\mathfrak{F}$ is said to be $\mathfrak{U}$-*small* if for each $x\in X$ there is a $U\in\mathfrak{U}$ containing $\{f(x)|f\in\mathfrak{F}\}$. Members of $\mathfrak{F}$ are said to be $\mathfrak{U}$-*approximate*. A homotopy $F:X\times I\to Y$ is said to be $\mathfrak{U}$-*small* if $\mathfrak{F}=\{F(\ ,t)|t\in I\}$ is. A function $g:X\to Y$ is said to be *approximated*

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by the set $\mathfrak{F}$ if for each $\mathfrak{U}$ there is a member f of $\mathfrak{F}$ such that f and g are $\mathfrak{U}$-approximate.

(3) Two embeddings $f, g: X \to Y$ are *ambient invertibly isotopic* if there is a level-preserving homeomorphism (called an invertible isotopy) $h: Y \times I \to Y \times I$ such that $h(y, 0) = (y, 0)$ for each $y \in Y$, and $h(f(x), 1) = (g(x), 1)$ for each $x \in X$.

(4) A pair (M, N) is an *$(l_2\ l_2^f)$-manifold pair* if M is an l_2-manifold for which there is an open cover $\mathfrak{U}$ and open embeddings $\{f_U: U \to l_2 \mid U \in \mathfrak{U}\}$ such that for each $U \in \mathfrak{U}$, $f_U(U \cap N) = f_U(U) \cap l_2$.

THEOREM 1. *If (M, N) is an (l_2, l_2^f)-manifold pair, then the inclusion of N into M is a homotopy equivalence.*

PROOF. Lemma 4 and Theorem 6 of [**13**] (which contains a topological characterization of (l_2, l_2^f)-manifold pairs) ensure that it induces isomorphisms on all homotopy groups, since any map of a sphere into M or N is homotopic to an embedding; and because M and N are ANR's for metric spaces it must be a homotopy equivalence (see [**9**]).

THEOREM 2 (UNIQUENESS OF (l_2, l_2^f)-MANIFOLD PAIRS). *Let M be an l_2-manifold, and let (M, N) and (M, N') be two (l_2, l_2^f)-manifold pairs. The identity map of (M, N) is approximated by homeomorphisms of the pair (M, N) onto (M, N') which are ambient isotopic to it by $\mathfrak{U}$-small isotopies for all $\mathfrak{U}$.*

PROOF. This is part of Theorem 6 of [**13**].

COROLLARY 1. *(l_2, l_2^f) is homeomorphic as a pair to $((l_2)^\omega, (l_2^f)_f^\omega)$.*

PROOF. This is the relative statement of Corollary 3 of [**13**]. The proof given there, however, coupled with Theorem 2 gives the result.

Note that l_2^f (or $(l_2^f)_f^\omega$) satisfies the hypotheses of Theorems 3–12.

THEOREM 3. (HOMEOMORPHISM SPACES ARE CONTRACTIBLE). *Let F be a TVS which is homeomorphic to F^ω or F_f^ω, then the space (C-O topology) of homeomorphisms of F is contractible.*

PROOF. If $F \cong F^\omega$ this is proved by Renz in [**10**], but an inspection of Renz's proof shows that it works with F^ω replaced by F_f^ω. (Note that, if $F \cong F_f^\omega$, then $F \cong F \times F$.)

THEOREM 4 (MICRO-BUNDLES ARE TRIVIAL). *Let F be a TVS which is homeomorphic to F^ω or F_f^ω and let B be a paracompact space with the homotopy-type of a simplicial (or CW) complex. Then every micro-bundle with base B and fiber F is trivial.*

PROOF. If $F \cong F^\omega$ this is proved by Henderson in [6]. The only place in [6] in which the hypothesis $F \cong F^\omega$ is used is in Lemma 4.2 which is proved in Renz's paper [10]. As above an inspection of Renz's proof shows that the hypothesis $F \cong F^\omega$ may be replaced by $F \cong F_f^\omega$.

THEOREM 5 (MANIFOLDS ARE STABLE). *Let F be a metrizable TVS which is homeomorphic to F^ω or F_f^ω. If M is a manifold modeled on F, then $M \times F \cong M$.*

PROOF. This theorem was proved by R. Schori in [11, Theorem 5.4] when $F \cong F^\omega$. As above, for $F \cong F_f^\omega$ apply the arguments of the proof of Theorem 5.4 of [11] to the TVS, F, and note that all maps involving F^ω can be restricted to F_f^ω. Otherwise, Schori's proof goes through without change.

THEOREM 6 (CLOSED EMBEDDING THEOREM). *Let F be a metrizable TVS which is homeomorphic to F^ω or F_f^ω. If M is a connected manifold modeled on F, then M can be embedded as a closed submanifold of F.*

PROOF. The proof is the same as the proof of Theorem 1 of [6], except that when $F \cong F_f^\omega$ one must note that, since $\{V_\alpha\}$ is locally finite, $\bar{h}_i(m)$ is zero for all but finitely many i's. Therefore h is an embedding

$$h: M \to (F \times F)_f^\omega \cong F.$$

THEOREM 7 (OPEN EMBEDDING THEOREM). *Let F be a metrizable locally-convex TVS (MLCTVS) which is homeomorphic to F^ω or F_f^ω. Each connected manifold, M, modeled on F can be embedded as an open subset of F.*

PROOF. Follows immediately from Theorem 5 of this paper and Theorem 2 of [7], because Theorem 4 of this paper allows the hypothesis, "$F \cong F^\omega$," of Theorem 2 of [7] to be changed to "$F \cong F_f^w$".

COROLLARY 2. *For M and F as in Theorem 7, there is an open cover $\mathfrak{U}$ of M such that any two $\mathfrak{U}$-approximate maps are homotopic by a $\mathfrak{U}$-small homotopy.*

THEOREM 8. *Let F and M be as in Theorem 7. Then the projection $p_1: M \times F \to M$ can be approximated by homeomorphisms $h: M \times F \to M$.*

PROOF. This follows from Schori's proof of Corollary 2.3 of [11] with the same modification as in the proof of Theorem 5.

THEOREM 9 (APPROXIMATING MAPS BY EMBEDDINGS). *Let M and N be manifolds modeled on a MLCTVS F which is homeomorphic to F^ω*

or F_f^ω. *Then each map* $f: M \to N$ *can be approximated by closed embeddings* $h_1: M \to N$ *and open embeddings* $h_2: M \to N$.

PROOF. Same as the proof of Theorem B of [8] because as noted above Theorem H3 of [8] which is Theorem 2 of [6] applies in this case.

THEOREM 10 (AMBIENT ISOTOPY THEOREM). *Let* N *be a manifold modeled on the normed TVS*, F, *which is homeomorphic to* F^ω *or* F_f^ω. *If* $f, g: X \to N \times \{0\} \subset N \times F$ *are homotopic closed embeddings of an ANR (for metric spaces), then* f *and* g *are ambient invertibly isotopic in* $N \times F$.

The proof is the same as Theorem 3 of [7] with obvious modifications as above.

THEOREM 11 (CLASSIFICATION BY HOMOTOPY TYPE). *Let* F *be as in Theorem* 10 *and let* M *and* N *be manifolds modeled on* F. *Then each homotopy equivalence between* M *and* N *is homotopic to a homeomorphism.*

The proofs of Theorems 11 and 12 are the same as Theorems C and D of [8] with modifications as above. Following R. D. Anderson we say that a subset K of a space X has *Property Z* in X if, for each nonempty, homotopically-trivial open set $U \subset X$, $U - K$ is nonempty and homotopically trivial.

THEOREM 12 (Z-SETS ARE NEGLIGIBLE). *Let* F *and* N *be as in Theorem* 10. *If* K *is a closed set with Property Z in* N, *then* K *is negligible, that is,* $N - K$ *is homeomorphic to* N. *In fact, the homeomorphism is homotopic to the inclusion* $N - K \to N$.

THEOREM 13. *If* K *is a locally finite-dimensional simplicial complex with the barycentric metric such that no star contains more than* $\mathfrak{A}$ *vertices, then* $(K \times l_2(\mathfrak{A}), K \times l_2^f(\mathfrak{A}))$ *is an* $(l_2(\mathfrak{A}), l_2^f(\mathfrak{A}))$*-manifold pair.*

PROOF. See [13, Corollary 6].

THEOREM 14 (COMPLETIONS OF l_2^f-MANIFOLDS). *If* N *is a manifold modeled on* l_2^f, *then there is an* (l_2, l_2^f)*-manifold pair* (M, N); *it is unique up to homeomorphisms of pairs.*

PROOF. N must have the homotopy type of a simplicial complex K satisfying the hypotheses of the above theorem (see [12, Theorem 5], or [6, Lemma 3.1]), so $(K \times l_2, K \times l_2^f)$ is an (l_2, l_2^f)-manifold pair in which $K \times l_2^f$ is homeomorphic to N by Theorem 11. The uniqueness is Theorem 1 followed by Theorem 11 (applied to the completion) followed by Theorem 2.

THEOREM 15 (TRIANGULATION OF l_2^f-MANIFOLDS). *Each l_2^f-manifold is homeomorphic to a metric simplicial complex.*

PROOF. As in the proof of Theorem 14, each l_2^f-manifold is homeomorphic to $K\times l_2^f$, for some metric simplicial complex K, but it is not difficult to show that l_2^f has a triangulation as a metric simplicial complex and (see [2]) that products of metric simplicial complexes are metric simplicial complexes.

DEFINITION. A *metric direct limit space* (*system*) is a direct limit space (system) in the category of metric spaces and isometries. Each metric space is the metric direct limit of any cover of itself which is closed under finite unions (with the maps being the inclusions).

THEOREM 16. *If $M^1\subset M^2\subset \cdots \subset M^n \cdots$ is a sequence of metrizable manifolds* ($\dim(M^n)=n$) *without boundary, each bicollared in the next, then the manifolds may be metrized so that it is a metric direct system whose limit is an $l_2^f(\aleph_0)$-manifold of the same homotopy type as the weak direct limit.*

REMARK. It is easy to construct such a sequence which has some metrics with respect to which the metric direct limit does not have the homotopy type of any $l_2^f(\aleph_0)$-manifold.

PROOF. An examination shows that the proof of 5.3 on pp. 186–187 of [3] establishes that any equivalent metric on a closed subset of a metric space may be extended to an equivalent metric on the entire space. An inductive application of this metrizes M^{n+1} so that the inclusion i_n of M^n, regarded as $M^n\times\{0\}$, into M^{n+1} extends to an isometric embedding i_n of $M^n\times(-1, 1)$. Then $M^1\subset M^2\subset \cdots \subset M^n \cdots$ is a metric direct system whose limit M is an $l_2^f(\aleph_0)$-manifold. (This may be seen since if $x\in M^n$ and U is an open neighborhood of x homeomorphic to R^n, then $U\times\{(x_1, \cdots)\in l_2^f \mid |x_i|<1, \text{ for all } i\}$ is isometric with inj $\lim\{i_{n+m}(U\times(-1, 1)_1\times \cdots \times(-1, 1)_m\}$, a neighborhood of inj $\lim\{x\}$ in M.)

To see that M is of the same homotopy type as M_w, the weak direct limit, it is only necessary to show that M and M_w have the same weak homotopy type, since they are ANR's. (See [9].) Regarding M and M_w as two different topologizations of the same underlying set and observing that the image of any map f of a compact space into M lies in one of the M^n's together with its successive collars (so that f is homotopic, relative to M^n, to a map into M^n), it is easy to see that the identity map of M_w to M is a weak homotopy equivalence.

THEOREM 17. *Each separable $l_2^f(\aleph_0)$-manifold M is the metric direct limit of a sequence $M^1\subset M^2\subset \cdots \subset M^n \cdots$ of closed, orientable manifolds* ($\dim(M^n)=n$) *each bicollared in its successor.*

PROOF. M has the homotopy type of a countable, locally finite, simplicial complex K. (For example, see [5].) Let K be embedded in $l_2^f(\aleph_0)$ simplicially by sending vertices to points of the standard orthonormal basis, and let L be a triangulation (with the norm-induced topology) of $l_2^f(\aleph_0)$ such that K and each R^n (the span of the first n basis elements) is a subcomplex. Define N and ∂N to be the subcomplexes of L'' (the second barycentric subdivision of L) composed respectively of all simplices which are faces of simplices meeting K and all such simplices which do not themselves meet K.

By standard regular neighborhood theory, $N \cap R^n$ and $\partial N \cap R^n$ are manifolds bicollared in $N \cap R^{n+1}$ and in $\partial N \cap R^{n+1}$, $\partial N \cap R^n$ is closed and orientable, K is a deformation retract of N, and ∂N is a deformation retract of $N-K$. Also, it may be shown directly or through [4] that $N-K$ is of the same homotopy type as N, so ∂N has the homotopy type of M. Dowker's theorem [2] then asserts that this is true of ∂N under the weak topology. By Theorem 16, ∂N may be remetrized to give an $l_2^f(\aleph_0)$-manifold, homotopy equivalent to M, which is the metric direct limit of $\{\partial N \cap R^n\}_{n=1}^{\infty}$, and by Theorem 11 it is homeomorphic to M.

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