ON THE SOLUTIONS OF THE NONLINEAR EIGENVALUE PROBLEM $Lu+\lambda b(x)u=g(x, u)$

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In connection with a problem in nonlinear reactor statics, we consider eigenvalue problems of the general form:

$$Lu + \lambda b(x)u = g(x, u), \quad x \in D, \tag{1}$$

$$\beta(x)\partial u/\partial \nu + \alpha(x)u = 0, \qquad x \in \partial D. \tag{2}$$

Here we take $x=(x_1, x_2, \cdots, x_m)$ and

$$\left.\begin{array}{l} L\phi \equiv \displaystyle\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\phi\right] - a_0(x)\phi, \quad a_{ij}(x) = a_{ji}(x), \\ \displaystyle\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geqq a^2 \sum_{i=1}^{m} \xi_i,\ a^2 > 0,\ a_0(x) \geqq 0, b\,(x) > 0, \end{array}\right\} x \in D; \tag{3}$$

$$\left.\begin{array}{l} \dfrac{\partial\phi}{\partial\nu} \equiv \displaystyle\sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\frac{\partial}{\partial x_j}\phi, \qquad \alpha(x)\beta(x) \geqq 0, \\ \alpha(x) \not\equiv 0, \alpha(x) + \beta(x) > 0, \end{array}\right\} x \in \partial D.$$

In addition, the functions $a_{ij}(x)$ and their derivatives are continuous on $\bar{D}$; the boundary is piecewise smooth with exterior unit normal $n(x)=(n_1(x), n_2(x), \cdots, n_m(x))$ for $x \in \partial D$. $g(x, u)$ is an analytic function of u.

The following lemma, which is established in substance by A. Hammerstein [1], is taken without proof here.

LEMMA. *Let the implicit equation,*

$$\sum_{m=2}^{\infty} L_{m0}\epsilon^m + \sum_{m=0}^{\infty} \epsilon^m \sum_{l=1}^{\infty} \mu^l L_{ml} = 0 \tag{4}$$

involving the small parameters ϵ *and* μ, *be such that the coefficients* L_{01} *and* L_{20} *are nonzero. Then there are exactly two solutions of* (4), *given by* $\epsilon(\mu) = \pm(L_{01}\mu/L_{20})^{1/2}[1+w(\mu)]$, *where* $w(\mu)\to 0$ *as* $\mu\to 0$.

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Note that ϵ is real only when $\operatorname{sgn}(\bar{\lambda} L_{01}/L_{20}) = \operatorname{sgn} \mu$. Equation (4) is called the "branch equation".

DEFINITION 1. The "branch point" of (1)–(2) is that value of λ, say λ_0, for which there is a solution of (1)–(2), say $u_0(x)$, and such that λ_0 is the principal eigenvalue of

$$
\begin{aligned}
L\phi + \lambda_0 b(x)\phi &= g_u(x, u_0(x))\phi, && x \in D, \\
\beta(x)\partial\phi/\partial\nu + \alpha(x)\phi &= 0, && x \in \partial D.
\end{aligned} \tag{5}
$$

Since λ_0 is the principal (least) eigenvalue of (5), we can insure, according to E. L. Ince [2], that $\phi(x)$ is of one sign in D. This follows since $\phi(x)$ has no zeros in D. We can take $\phi(x)$ to be positive without loss of generality.

The integral equation equivalent to (5) is

$$
\phi(x) = \int_D K(x; t)[\lambda_0 b(t) - g_u(t, u_0(t))]\phi(t)dt, \tag{6}
$$

where $K(x; t)$, $x \in D$ and $t \in D$, is the kernel corresponding to the operator L.

$\phi^+(x)$ is the eigenfunction, corresponding to λ_0, of the transposed kernel $K(t; x)[\lambda_0 b(x) - g_u(x, u_0(x))]$, and for symmetric kernels $K(x; t) \equiv K(t; x)$ is equal to

$$
\phi^+(x) = \sigma[\lambda_0 b(x) - g_u(x, u_0(x))]\phi(x). \tag{7}
$$

This can be shown by multiplying (6) by $[\lambda_0 b(x) - g_u(x, u_0(x))]$; here $\sigma^2 \equiv 1/\int_D [(\lambda_0 b - g_u)\phi]^2 dx$ is a normalization constant, where the denominator, equal to $\int_D [L\phi]^2\, dx$, is taken to be nonvanishing.

We point out that the conditions of Definition 1 may not be met by any value of λ; in this case we say that there is no branch point.

In order to investigate the existence and uniqueness of solutions for λ in a small neighborhood of λ_0, denoted by $\lambda \in \mathfrak{n}(\lambda_0)$, we set $v \equiv u - u_0$, $\mu \equiv \lambda - \lambda_0$ and $\epsilon = \int_D \phi v dx$. After some effort an associated branch equation is obtained where the first few coefficients are given by

$$
L_{01} = \int_D dx \int_D dt K(x; t) u_0(t) \phi^+(x); \tag{8}
$$

$$
L_{20} = \frac{1}{2} \int_D dx \int_D dt[-K(x; t) g_{uu}(t, u_0(t)) \phi^2(t) \phi^+(x)]. \tag{9}
$$

In this development an expansion for $v(x; \mu)$ of the form

$$
v(x; \mu) = \epsilon v_{10}(x) + \mu v_{01}(x) + \epsilon^2 v_{20}(x) + \mu\epsilon v_{11}(x) + \cdots \tag{10}
$$

is utilized, and $v_{10}(x)$ is found to be just equal to $\phi(x)$.

We are now prepared for our first result.

THEOREM 1. *Suppose that the branch point* λ_0 *exists for some* $u_0(x)>0$. *Furthermore, suppose* $\lambda_0 b(x)-g_u(x, u_0(x))>0$ *for all* $x\in D$, *so that* $\phi^+(x)>0$. *Then for all operators* L *possessing positive kernels* $K(x; t)$

(a) *if* $g_{uu}(x, u_0(x))<0$, *there are exactly two positive solutions to* (1)–(2) *for* $\lambda<\lambda_0$ *and no solution for* $\lambda>\lambda_0$, *when* $\lambda\in\mathfrak{N}(\lambda_0)$,

(b) *if* $g_{uu}(x, u_0(x))>0$, *there are exactly two positive solutions to* (1)–(2) *for* $\lambda>\lambda_0$ *and no solution for* $\lambda<\lambda_0$, *when* $\lambda\in\mathfrak{N}(\lambda_0)$. *In both cases* (a) *and* (b) *the two solutions,* $u_1(x; \lambda)$ *and* $u_2(x; \lambda)$, *are such that* $u_1(x; \lambda)\leqq u_0(x)\leqq u_2(x; \lambda)$.

PROOF. We give details for (a). Under the conditions given, the integrands of the expressions (8) and (9) are positive in D, hence $\operatorname{sgn} L_{01}=\operatorname{sgn} L_{20}$. Then by the lemma there are exactly two real solutions to the associated branch equation when $\mu<0$, i.e., $\lambda<\lambda_0$, and no real solution when $\lambda>\lambda_0$. Furthermore, as $\mu\to 0$, that is, for $\lambda\in\mathfrak{N}(\lambda_0)$, $\epsilon(\mu)\to\pm(-L_{01}\mu/L_{20})^{1/2}$. Hence for $\lambda<\lambda_0$, by equation (10) and the definition of $v(x; \mu)$,

$$(11)\quad u(x; \lambda) = u_0(x) \pm (-L_{01}(\lambda-\lambda_0)/L_{20})^{1/2}\phi(x), \qquad \lambda\in\mathfrak{N}(\lambda_0).$$

The two solutions are clearly positive, and greater than and less than $u_0(x)$, respectively, for $\mathfrak{N}(\lambda_0)$ sufficiently small. The proof of (b) is similar.

Thus the theorem is proved, and an expression for the two solutions for $\lambda\in\mathfrak{N}(\lambda_0)$ is derived. ∎

The above theorem remains valid if the condition $\lambda_0 b(x)-g_u(x, u_0(x))>0$ is omitted to give a more general result; however, the neat condition in (a), $g_{uu}(x, u_0(x))<0$, must be replaced by $L_{20}>0$, and in (b), $g_{uu}(x, u_0(x))>0$ must be replaced by $L_{20}<0$.

The expression (11) for $u(x; \lambda)$ indicates that the intersection of the solution surface with any plane $x=x_0$, $x_0\in D$, is clearly a parabola with vertex at $\lambda=\lambda_0$, $u=u_0$, when $\lambda\in\mathfrak{N}(\lambda_0)$.

It is of interest to obtain an estimate of the branch point for problems which exhibit the features described by Theorem 1. This work has connection with previous work by D. Joseph [3].

THEOREM 2. *Suppose* Λ_0 *is the principal eigenvalue of the Helmholtz equation*

$$(12)\quad \begin{aligned} L\psi + \Lambda b(x)\psi &= 0; & x&\in D,\\ \beta(x)\partial\psi/\partial\nu + \alpha(x)\psi &= 0, & x&\in\partial D. \end{aligned}$$

Then λ_0 is bounded above for case (a) in Theorem 1 by $\mathrm{ub}(\lambda_0)$, and below for case (b) by $\mathrm{lb}(\lambda_0)$, where

$$\mathrm{ub}(\lambda_0) = \max_{\eta \geqq 0, x \in \bar{D}} \left[\Lambda_0 + \frac{g(x,\eta)}{b(x)\eta} \right], \tag{13}$$

$$\mathrm{lb}(\lambda_0) = \min_{\eta \geqq 0, x \in \bar{D}} \left[\Lambda_0 + \frac{g(x,\eta)}{b(x)\eta} \right]. \tag{14}$$

PROOF. To show this we combine (1)–(2), (12) and Green's first formula for L to give

$$\begin{aligned} 0 &= \int_D [uL\psi_0 - \psi_0 Lu] dx \\ &= \int_D \psi_0(x) g(x, u(x)) dx + \Lambda_0 \int_D u(x) b(x) dx - \lambda \int_D \psi_0(x) u(x) b(x) dx \end{aligned}$$

where $\psi_0(x)$ is the eigenfunction associated with Λ_0 and use is made of the selfadjointness of the operator L and its associated boundary conditions. We assume $\int_D uL\psi_0 dx \neq 0$.

Then

$$\lambda = \frac{\int_D [\Lambda_0 + g(x, u(x))/b(x)u(x)] b(x)\psi_0(x)u(x)dx}{\int_D b(x)\psi_0(x)u(x)dx}$$

where the denominator is nonzero, as can be demonstrated by multiplying (12) by $u(x)$, integrating over D and using the fact that $\int_D uL\psi_0 dx \neq 0$.

Thus

$$\min_{\eta \geqq 0, x \in \bar{D}} \left[\Lambda_0 + \frac{g(x,\eta)}{b(x)\eta} \right] \leqq \lambda \leqq \max_{\eta \geqq 0, x \in \bar{D}} \left[\Lambda_0 + \frac{g(x,\eta)}{b(x)\eta} \right], \tag{15}$$

which gives the bounds on λ_0 stated above, for the cases described in Theorem 1. ■

Note that the inequality (15) is quite general, but acquires significant meaning for the specific functions $g(x, u)$ whose properties result in the cases described in Theorem 1. That is, for these cases the inequality provides upper and lower bounds for the spectrum of (1)–(2).

It is also possible to obtain an "a priori" estimate of $\max_{x \in \bar{D}} \{u_0(x)\}$ in the process of computing the bound on λ_0; namely, $\max_{x \in \bar{D}} \{u_0(x)\}$

$=\eta$, where η maximizes or minimizes the expression $[g(x, \eta)/b(x)\eta]$ for $\eta>0$ and $x\in\bar{D}$.

DEFINITION 2. A solution $u(x; \lambda)$ is said to be a "stable" solution if $\lim_{t\to\infty} u^*(t, x; \lambda)$ exists and is equal to $u(x; \lambda)$, where $u^*(t, x; \lambda) \equiv u(x; \lambda)+w(x; \lambda)e^{-ct}$ is the solution to the related time-dependent problem,

$$\begin{aligned} &\partial u^*/\partial t + Lu^* + \lambda b(x)u^* = g(x, u^*), && x \in D, \quad t > 0, \\ &\beta(x)\partial u^*/\partial\nu + \alpha(x)u^* = 0, && x \in \partial D, \quad t > 0. \end{aligned} \tag{16}$$

The "stability" of the two solutions described by Theorem 1 has been investigated, and it has been determined that for functions $g(x, u)$ which are described by (a) of Theorem 1, the smaller solution is "stable", while the larger solution is "unstable". The reverse is true for functions described by (b). This result follows by the differentiation of (1)–(2) with respect to $P=\max_{x\in\bar{D}} \{u(x; \lambda)\}$, and examining the nature of c for $\lambda\in\mathfrak{N}(\lambda_0)$ using the resulting equation and (16).

Finally, both solutions can be obtained by a modification of a constructive procedure due to L. Shampine [4], provided $g(x, u)$ satisfies a growth condition,

$$g(x, \phi) - g(x, \psi) \leqq k(x)\cdot(\phi - \psi) \quad \text{for } \phi \geqq \psi. \tag{L}$$

THEOREM 3. *Suppose there are twice continuously differentiable functions $y(x)$, $Y(x)$ such that $y(x)\leqq Y(x)$ for $x\in D$ and*

$$\begin{aligned} &Ly(x) + \lambda b(x)y(x) \geqq g(x, y(x)), && x \in D, \\ &\beta(x)\partial y/\partial\nu + \alpha(x)y = 0, && x \in \partial D, \\ &LY(x) + \lambda b(x)Y(x) \leqq g(x, Y(x)), && x \in D, \\ &\beta(x)\partial Y/\partial\nu + \alpha(x)Y = 0, && x \in \partial D. \end{aligned}$$

Suppose $g(x, \phi)$ is continuous and satisfies (L) *on the set*

$$S = [(x, \phi) \,|\, x \in D, y(x) \leqq \phi \leqq Y(x)].$$

Let N denote a nonnegative integer, 0, 1, 2 $\cdots$, *and let Λ_0 be the principal eigenvalue of* (12). *For any λ in $N\Lambda_0\leqq\lambda\leqq(N+1)\Lambda_0$ and for suitable $z_0(x)$, define the sequence $[z_n(x)]$ by*

$$\begin{aligned} Lz_n(x) + (\lambda - N\Lambda_0)b(x)z_n(x) &= g(x, z_{n-1}(x)) \\ &\quad + k(x)[z_n(x) - z_{n-1}(x)] - N\Lambda_0 b(x)z_{n-1}(x), \qquad x \in D, \\ \beta(x)\partial z_n/\partial\nu + \alpha(x)z_n &= 0, \qquad x \in \partial D. \end{aligned} \tag{17}$$

Then if $z_0(x)=y(x)$, the sequence converges monotonely and uniformly upwards to the least solution of (1)–(2) *which lies above $y(x)$ (and below*

$Y(x)$). *If* $z_0(x) = Y(x)$, *the sequence converges monotonically and uniformly downwards to the greatest solution of* (1)–(2) *which lies below* $Y(x)$ (*and above* $y(x)$).

The proof of this theorem follows arguments similar to those utilized both by Shampine and by D. Cohen [5], who treats a problem similar to (1)–(2); these arguments are omitted here.

We note that instead of $k(x)$, one can use $g_u(x, z_{n-1}(x))$ in the constructive procedure (17), provided the first partial derivative exists and satisfies (L). The advantage is one of more rapid convergence.

Two examples have been studied as illustrations of the above results, and both branches of the parabolic curve traced out by (11) in the plane $x = x_0$, $x_0 \in \overline{D}$, where $u(x_0) = \max_{x \in \overline{D}} \{u(x; \lambda)\}$ have been obtained by the iterative formula (17). These will be published in a future paper, along with details for the above results.

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