COMMUTATIVE RINGS WITH IDENTITY HAVE RING TOPOLOGIES

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Communicated by Leonard Gillman, October 16, 1969

Throughout, let R denote a commutative ring with identity. By a *proper* topology we mean a separated nondiscrete ring topology.

THEOREM. *Every infinite R has a proper topology.*

The following five propositions outline the proof. Details will appear in [**2**].

PROPOSITION 1. *Each infinite R satisfies at least one of the following conditions.*

(a) *R admits a proper ideal topology (i.e. one having a neighborhood basis at 0 consisting of ideals).*

(b) *R contains infinitely many nilpotents.*

(c) *There is an element $r \in R$ such that $R/\text{Ann}_R r$ is an infinite field.*

The proof depends on the characterizations of rings with proper ideal topologies in [**1**]. To prove the theorem we now need only consider rings satisfying (b) or (c).

PROPOSITION 2. *Let I be an ideal of R having a proper R-algebra topology $\mathfrak{J}$ (R discrete). There is a unique proper topology on R such that I (with topology $\mathfrak{J}$) is an open subspace.*

PROPOSITION 3. *Let $\phi: R/\text{Ann}_R r \to (r)$ be the obvious R-module isomorphism. For each proper topology on $R/\text{Ann}_R r$, ϕ induces a proper R-algebra topology on (r). Hence, if $R/\text{Ann}_R r$ has a proper topology, so does R.*

It is known that all infinite fields have proper topologies [**3**, Theorem 5.2, p. 159]. With this result and Proposition 3 we have

COROLLARY. *If R satisfies* (c), *R has a proper topology.*

PROPOSITION 4. *Every infinite abelian group I has a separated nondiscrete group topology such that every endomorphism of I is continuous.*

AMS Subject Classifications. Primary 1698, 1350; Secondary 1270, 1320, 2210.

Key Words and Phrases. Topological ring, commutative ring with identity, existence of ring topologies, ideal topology.

[1] Both authors were supported in part by NSF Grant GP-8496.

(The proof is an easy consequence of Pontrjagin duality.)

PROPOSITION 5. *Let R satisfy* (b). *Then there is an infinite ideal I of R such that $I^2 = (0)$. If $\mathfrak{I}$ is an additive group topology for I as in Proposition* 4, *then $\mathfrak{I}$ is a proper R-algebra topology for I. Hence, by Proposition* 2, *R has a proper topology.*

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