INVERSE LIMITS AND MULTICOHERENCE

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1. **Introduction.** In this paper we state some results concerning inverse limits and multicoherence and give applications to hyperspaces and inverse limits of special types of spaces. The proofs of these and other related results will appear elsewhere.

By a *metric continuum* we mean a nonempty compact connected metric space. A space X is said to have *property* (b) (see [2, p. 63] or [10, p. 226]) if and only if, given a continuous function f from X into the unit circle in the plane, there exists a continuous real valued function α defined on X such that $f(x) = e^{i\alpha(x)}$ for each $x \in X$. If X is a metric continuum, then we say that X has *multicoherence degree k* (see [3] or [10, p. 83]) provided l.u.b. $\{r(X_1, X_2): X_1$ and X_2 are subcontinua of X with $X_1 \cup X_2 = X\} = k$, where $r(X_1, X_2)$ denotes one less than the number of components of $X_1 \cap X_2$. The multicoherence degree of X is denoted by $r(X)$. We note that $r(X) = 0$ is equivalent to X being unicoherent. It is well known that if X is a metric continuum with property (b), then X is unicoherent (see [2, p. 69] or [10, p. 227]), but not conversely.

All inverse systems considered in this paper are countable and the inverse limit of an inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$ is denoted by **proj lim** $\{X_i, f_i\}_{i=1}^{\infty}$. For notation and terminology relating to inverse limits, see [1].

2. **Basic theorems.**

THEOREM 1. *If* $X =$ proj $\lim \{X_i, f_i\}_{i=1}^{\infty}$ *and each space* X_i *is a metric continuum with property* (b), *then* X *has property* (b).

THEOREM 2. *If* $X =$ proj $\lim \{X_i, f_i\}_{i=1}^{\infty}$ *where, for each* $i = 1, 2, \cdots, X_i$ *is a metric continuum,* $r(X_i) \leq k$, *and* f_i *is a mapping of* X_{i+1} *onto* X_i, *then* $r(X) \leq k$.

We note that if each of the bonding maps f_i in Theorem 2 are monotone and $r(X_i) = k$ for all $i = 1, 2, \cdots$, then $r(X) = k$. This observation leads to very simple proofs of Theorem 4.8 and Theorem 4.11 of [1].

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COROLLARY 1. *If* $X = \text{proj lim}\{X_i, f_i\}_{i=1}^{\infty}$ *where* X_i *is a unicoherent metric continuum and* f_i *maps* X_{i+1} *onto* X_i *for each* $i = 1, 2, \cdots$, *then* X *is unicoherent.*

COROLLARY 2. *Let* $X = \text{proj lim}\{X_i, f_i\}_{i=1}^{\infty}$ *where each space* X_i *is a compact metric space. If every subcontinuum of* X_i *has property* (b) (*has multicoherence degree* $\leq k$) *for all* $i = 1, 2, \cdots$, *then every subcontinuum of* X *has property* (b) (*has multicoherence degree* $\leq k$).

Theorem 1 states that property (b) is preserved by taking inverse limits (with no restrictions on the bonding maps). Since property (b) is equivalent to unicoherence for locally connected metric continua (see [2] or [**10**, p. 228]), it follows from Theorem 1 that the inverse limit of locally connected unicoherent metric continua is unicoherent (note that the inverse limit space might not be locally connected). However, in general, an inverse limit of unicoherent metric continua need not be unicoherent. The following example illustrates this.

EXAMPLE. For each $i = 1, 2, \cdots$, let $X_i = S^1 \cup \mathcal{S}$ where S^1 is the unit circle in the plane and $\mathcal{S} = \{((t+1)/t)e^{2\pi i t} : 1 \leq t < \infty\}$. Also, for each $i = 1, 2, \cdots$, let $r_i : X_{i+1} \to X_i$ be given by $r_i(z) = z/|z|$ for all $z \in X_{i+1}$. It is easy to verify that the space $\text{proj lim}\{X_i, r_i\}_{i=1}^{\infty}$ is homeomorphic to S^1 but that each space X_i is unicoherent.

3. **Applications.** In this section we state some results on hyperspaces and inverse limits of special types of continua. Their proofs utilize material in the previous section.

Let $2^X = \{K : K \text{ is a nonempty compact subset of } X\}$ and let $C(X) = \{K : K \in 2^X \text{ and } K \text{ is connected}\}$. The topology for 2^X is the finite topology [7] ($C(X)$, as a subset of 2^X carries the relative topology from 2^X). Known theorems (see, for example, [**5**], [**6**], [**9**], or [**11**]) indicate a general pattern that these hyperspaces are less pathological than the space. Our next theorem shows that if X is any metric continuum, then 2^X and $C(X)$ have property (b) and therefore are unicoherent. Of course, if X is locally connected, this is a simple consequence of the result in [**11**].

THEOREM 3. *If* X *is a metric continuum, then both* 2^X *and* $C(X)$ *have property* (b).

A *dendroid* is an arcwise connected metric continuum such that each subcontinuum is unicoherent. The following corollary extends Lemma 3 of [**8**] which states that if a metric continuum admits a continuous selection [**7**] on its hyperspace of (nonempty) subcontinua, then it is a dendroid.

COROLLARY 3. *If a metric continuum* X *is a retract of* $C(X)$ (*we* *consider X contained in $C(X)$ as the subspace of singletons), then X is arcwise connected and has property* (b).

REMARK. It is clear that if X is a locally connected metric continuum, then X is a retract of 2^X (or of $C(X)$) if and only if X is an absolute retract. We pose the following question.

QUESTION. What are necessary and sufficient conditions in order that a metric continuum X be a retract of 2^X or of $C(X)$?

A *dendrite* [**10**, p. 88] is a locally connected metric continuum which contains no simple closed curve. A mapping $f: Y \to Z$ is said to be *monotone* if and only if $f^{-1}(f(y))$ is a continuum for each $y \in Y$ (see [**10**, p. 70] but note that we do not require f to be onto Z).

THEOREM 4. *Let $X = \text{proj lim}\{D_i, f_i\}_{i=1}^{\infty}$ where D_i is a dendroid for each $i = 1, 2, \cdots$.*

1. *If X is arcwise connected, then X is a dendroid.*
2. *If X is locally connected, then X is a dendrite.*
3. *If D_i is a dendrite and f_i is a monotone mapping of D_{i+1} onto D_i for each $i = 1, 2, \cdots$, then X is a dendrite.*

Some of the results in [4] are simple consequences of the theorems in this paper and some can be extended using these theorems. The next corollary extends the theorem in [4] which states that if the inverse limit (with onto bonding maps) of arcs is locally connected, then it is an arc.

COROLLARY 4. *If $X = \text{proj lim}\{A_i, f_i\}_{i=1}^{\infty}$ where A_i is an arc for each $i = 1, 2, \cdots$ and if X is arcwise connected, then X is an arc or a singleton.*

PROOF. By Theorem 4, X is a dendroid. If X is neither an arc nor a singleton, then X contains a triod T. Using 2.8 of [**1**, p. 235] we see that $T = \text{proj lim}\{\pi_i(T), f_i | \pi_{i+1}(T)\}_{i=1}^{\infty}$. Since $\pi_i(T)$ is an arc for each $i = 1, 2, \cdots$, we have a contradiction.

The results in §2 can be done in the setting of inverse systems over directed sets and general Hausdorff continua. Using this more general setting, Theorem 3 of §3 can be proved for X compact, connected, and Hausdorff.

It has been pointed out to me by Professor John Isbell that property (b) is equivalent to $H'(X, Z) = 0$ so that Theorem 1 is a consequence of the continuity of Čech theory.

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