ON THE CLASSIFICATION OF LIE ALGEBRAS OF PRIME CHARACTERISTIC[1]

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This paper presents results on the classification of finite-dimensional simple Lie algebras of prime characteristic. Unlike the simple algebras of characteristic zero, these algebras do not necessarily possess a nondegenerate trace form—an important tool in classifying the characteristic zero algebras. Those which do possess such a form have been classified by G. B. Seligman [9] and, not unexpectedly, have been found to be analogs of the simple algebras of characteristic zero. However, further large classes of a quite different nature exist. Having no characteristic zero analogs, they are said to be of nonclassical type. Important work on the classification of such algebras has been done by A. I. Kostrikin ([5] and a long series of earlier papers) and by Kostrikin and I. R. Šafarevič ([7] and [8]). In an attempt to provide a uniform classification theory, they have considered the infinite Lie algebras over the complex numbers (see [1], [2], [3]). These algebras are said to be of Cartan type and have been completely classified. By replacing the complex numbers in the construction by an algebraically closed field of prime characteristic, Kostrikin and Šafarevič have produced a unified way of describing all the known nonclassical simple restricted Lie algebras. This has been further generalized by R. L. Wilson [10] to include all the known simple algebras of nonclassical type. The fact that their construction exhausts the known simple restricted algebras has led Kostrikin and Šafarevič to conjecture that there are no more such algebras. This conjecture has been proved in a very special case [7, Theorem 1]. As has developed in their work, the classification problem splits into two parts: proving the existence of a long filtration and using the filtration in an attempt to classify the algebras. By a filtration of a finite-dimensional Lie algebra L is meant a chain of subalgebras

$$L = L_{-1} \supset L_0 \supset \cdots \supset L_r \supset L_{r+1} = 0,$$

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where L_0 is an arbitrary proper subalgebra (usually taken to be maximal) and the remaining subalgebras are defined inductively by $L_{i+1}=\{x\in L_i \mid [x, L]\subset L_i\}$. A filtration is said to be long if $r\geqq 2$.

The present paper is a contribution toward establishing the existence of a long filtration in the simple Lie algebras which are not of classical type. Kostrikin [5] has shown this to be equivalent to an algebra L being strongly degenerate, that is, possessing an element of index of nilpotence two. An element x in L with $L(\text{ad } x)^k=0$, $L(\text{ad } x)^{k-1}\neq 0$ is said to be nilpotent of index k. Kostrikin's main result in [5] is formulated as follows. If L is a simple Lie algebra over an algebraically closed field of characteristic $p>5$ which satisfies:

(i) L is restricted, that is, ad L is closed under Pth powers (restricted Lie algebras are also called Lie p-algebras),

(ii) L is not strongly degenerate,

(iii) L has a Cartan subalgebra H, $L=H+\sum_{\alpha\neq 0} L_\alpha$, for which $L(\text{ad } a)^{p-1}=0$ for some nonzero a in H or in some root space L_α, then L is of classical type.

If, as in (iii), an element a is either in L_α for some α or in H, it is called H-uniform. In addition to weakening (ii), we can show that deleting (i) does not alter the conclusion. Our first main result is the following.

THEOREM A. *Let L be a simple algebra over an algebraically closed field F of characteristic $p>5$. Let H be a Cartan subalgebra of L for which* (iii) *is satisfied, and further suppose that any nilpotent H-uniform element has index of nilpotence greater than two. Then L is of classical type.*

We remark that condition (iii) cannot be deleted from the hypothesis of Theorem A as the nonclassical p-dimensional Witt algebra possesses a Cartan subalgebra for which no uniform element is nilpotent.

This theorem is a contribution toward showing (as Kostrikin conjectures) that strong degeneration differentiates between algebras of classical and nonclassical type (it can be easily shown that no classical algebra is strongly degenerate).

We now consider some special algebras of low dimension relative to p. Suppose L is a simple Lie algebra over an algebraically closed field of characteristic $p>5$. Kostrikin has shown that L is of classical type if dim $L<p$, and he has some partial results [4] for dimension p. We use Theorem A and several results of Kostrikin to show that if L has dimension p, then it is either the nonclassical p-dimensional Witt algebra or of classical type (Theorem B). The final theorem, which proves a conjecture of Kostrikin in [6], is the following.

THEOREM C. *Suppose L is a semisimple Lie algebra over an algebraically closed field F of prime characteristic which admits a faithful representation Γ of degree $n<p-1$. Then L is a direct sum*

$$L = L_1 \oplus \cdots \oplus L_r,$$

where each L_i is a simple algebra of classical type.

Kostrikin has proved this under the additional hypotheses that L is a p-algebra and Γ is a p-representation.

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